CLASS DISCUSSION: 18 FEBRUARY 2019

CHAIN RULES

1. *Review*

Here we exploit the fact that the tangent plane is the *best linear approximation* to a surface at a given point.

(a) By *roughly how much* will $f(x, y) = x (x^2 + y^2)^{1/2}$ change as one moves from point P = (4, 3) a distance $\Delta s = 0.1$ *unit* in the direction of the vector $\mathbf{a} = 2\mathbf{i} - 3\mathbf{j}$? Compare this to the actual value of the change.

(b) By *roughly how much* will $g(x, y) = e^x \cos y$ change as one moves from the origin a distance $\Delta s = 0.2$ *unit* in the direction of the vector $\mathbf{b} = 5\mathbf{i} + 12\mathbf{j}$? Compare this to the actual value of the change.

2. Extend the tangent plane result to a function of 3 variables.

3. (Stewart)

The dimensions of a rectangular box are measured to be 75 cm, 60 cm, and 40 cm, and each measurement is correct to within 0.2 cm. Use differentials to estimate the largest possible error when the volume of the box is calculated from these measurements.

4. State the *special case* of the Chain Rule for $\frac{d}{dt}f(r(t))$.
Express this result in terms of the *gradient* of f.

5. Compute $\frac{d}{dt}$ $f(\mathbf{r}(t))$ for each of the following:

(a) $f(x, y) = xy$, $\mathbf{r}(t) = (e^t, \cos t)$

(b) $f(x, y) = e^{xy}$, $\mathbf{r}(t) = (3t^2, t^3)$

(c) $f(x, y) = x \exp(x^2 + y^2)$, $\mathbf{r}(t) = (t, -t)$

6. (From Marsden's **Vector Calculus**) Suppose that a swan is swimming in the circle $x = \cos t$ and $y = \sin t$ and that the water temperature is given by $T(x,y) = x^2e^y - xy^3$.
Find $\frac{dT}{dt}$, the rate of change in temperature the swan might experience: (a) by using the Chain Rule; (b) by expressing *T* in terms of *t* and differentiating.

7. Stewart:

The pressure P (in kilopascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. Find the rate at which the pressure is changing when the temperature is 300 K and increasing at a rate of 0.1 K/s and the volume is 100 L and increasing at a rate of 0.2 L/s.

8. Using the Chain Rule for a function of three variables, find dw/dt where $w(x, y, z) = x/z + y/z$, $x(t) = \cos^2 t$, $y(t) = \sin^2 t$, $z(t) = 1/t$, $t = 3$.

9. The second special case of the chain rule:

Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

7, 8, 9, 10, 11 and 12 Use the Chain Rule to find $\partial z/\partial s$ and $\partial z/\partial t$.

7. $z = (x - y)^5$, $x = s^2 t$, $y = st^2$

Answer

8. $z = \tan^{-1}(x^2 + y^2)$, $x = s \ln t$, $y = te^s$

9. $z = \ln(3x + 2y)$, $x = s \sin t$, $y = t \cos s$

Answer

10. $z = \sqrt{x} e^{xy}$, $x = 1 + st$, $y = s^2 - t^2$

11. $z = e^r \cos \theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$

Answer

12. $z = \tan(u/v)$, $u = 2s + 3t$, $v = 3s - 2t$

13. Let $p(t) = f(g(t), h(t))$, where f is differentiable, $g(2) = 4$, $g'(2) = -3$, $h(2) = 5$, $h'(2) = 6$, $f_x(4, 5) = 2$, $f_y(4, 5) = 8$. Find $p'(2)$.

Answer

14. Let $R(s, t) = G(u(s, t), v(s, t))$, where G, u, and v are differentiable, $u(1, 2) = 5$, $u_s(1, 2) = 4$, $u_t(1, 2) = -3$, $v(1, 2) = 7$, $v_s(1, 2) = 2$, $v_t(1, 2) = 6$, $G_u(5, 7) = 9$, $G_v(5, 7) = -2$. Find $R_s(1, 2)$ and $R_t(1, 2)$.

15. Suppose f is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin v, e^u + \cos v)$. Use the table of values to calculate $g_u(0, 0)$ *and* $g_v(0, 0)$.

	f	g	f_x	f_y
(0, 0)	3	6	4	8
(1, 2)	6	3	2	5

Answer

16. Suppose f is a differentiable function of x and y, and $g(r, s) = f(2r - s, s^2 - 4r)$. Use the table of values in Exercise 15 to calculate $g_r(1, 2)$ and $g_s(1, 2)$.

10. Using the special case of the Chain Rule, find (i) a formula for $\frac{d}{dt}e^{f(t)g(t)}$ and (ii) a formula for $\frac{d}{dt}f(t)^{g(t)}$.

11. Stewart

35. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds?

Answer

36. Wheat production W in a given year depends on the average temperature T and the annual rainfall R. Scientists estimate that the average temperature is rising at a rate of 0.15°C/year and rainfall is decreasing at a rate of 0.1 cm/year. They also estimate that at current production levels, $\partial W/\partial T = -2$ and $\partial W/\partial R = 8$.

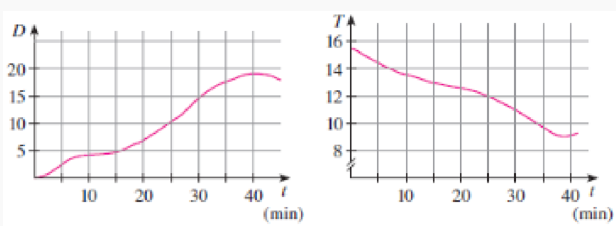
a. What is the significance of the signs of these partial derivatives?

b. Estimate the current rate of change of wheat production, dW/dt.

37. The speed of sound traveling through ocean water with salinity 35 parts per thousand has been modeled by the equation

$$C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$$

where C is the speed of sound (in meters per second), T is the temperature (in degrees Celsius), and D is the depth below the ocean surface (in meters). A scuba diver began a leisurely dive into the ocean water; the diver's depth and the surrounding water temperature over time are recorded in the following graphs. Estimate the rate of change (with respect to time) of the speed of sound through the ocean water experienced by the diver 20 minutes into the dive. What are the units?



38. The radius of a right circular cone is increasing at a rate of 1.8 in/s while its height is decreasing at a rate of 2.5 in/s. At what rate is the volume of the cone changing when the radius is 120 in. and the height is 140 in.?

39. The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 1$ m and $w = h = 2$ m, and ℓ and w are increasing at a rate of 2 m/s while h is decreasing at a rate of 3 m/s. At that instant find the rates at which the following quantities are changing.

   a. The volume

   Answer

   b. The surface area

   Answer

   c. The length of a diagonal

   Answer

40. The voltage V in a simple electrical circuit is slowly decreasing as the battery wears out. The resistance R is slowly increasing as the resistor heats up. Use Ohm's Law, $V = IR$, to find how the current I is changing at the moment when $R = 400\ \Omega$, $I = 0.08$ A, $dV/dt = -0.01$ V/s, and $dR/dt = 0.03\ \Omega$/s.

41. The pressure of 1 mole of an ideal gas is increasing at a rate of 0.05 kPa/s and the temperature is increasing at a rate of 0.15 K/s. Use the equation $PV = 8.31T$ in Example 2 to find the rate of change of the volume when the pressure is 20 kPa and the temperature is 320 K.

12. Find dz/dt in two different ways if $z = x/y$ and $x = \sin t$, $y = \cos t$.

13. Find dz/dt if $z = \ln(xy)$, $x = \exp(t)$, $y = \exp(-t)$.

14. Find dw/dt if $w = \exp(xyz)$, $x = t$, $y = t^2$, $z = t^3$.

15. Suppose that the temperature at each point of the plane is given by $T = (x^2 + y^2)^{1/2}$. The position of a bug at time *t* is given by $x = t^2$, $y = t^3$. Determine the *rate of change of the temperature* experienced by the bug as it passes through the point (4, 8).

16. Let $w = u^2 e^v$, $u = x/y$, $v = y \ln x$. Find w_x and w_y at $(x, y) = (1, 2)$.

17. If $z = (x - y)/(x + y)$ where $x = uvw$, $y = u^2 + v^2 + w^2$, determine z_u, z_v, z_w where $u = 2$, $v = -1$, and $w = 1$.

18. If $z = y/x$, $x = e^u \cos v$, $y = e^u \sin v$, determine z_u and z_v.

19. Let $u = x \exp(yz)$ and $(x, y, z) = (e^t, t, \sin t)$. Find du/dt in two different ways.

20. Let f (x, y) be given. Let $x = r\cos\theta$ and $y = r \sin\theta$. Find f_θ and f_r.

21. Let $f(x, y) = xy$ and $x = u^2 - v^2$ and $y = u^2 + v^2$. Find f_u and f_v.

22. Suppose that a duck is swimming in a straight line $x = 3 + 8t$, $y = 3 - 2t$ in the plane, while the water temperature is given by

$$T(x, y) = x^2 \cos y - y^2 \sin x.$$

(Assume that *x* and *y* are given in feet, *t* is given in minutes, and *T* is given in degrees Fahrenheit.) Find dT/dt.

23. Suppose that the temperature at (x, y, z) in 3-space is given by $T(x, y, z) = x^2 + y^2 + z^2$. Assume that a particle moves along a right circular helix $\boldsymbol{\sigma}(t) = (\cos t, \sin t, t)$. Let T(t) be its temperature at time *t*.

(a) Find dT/dt.

(b) Find an approximate value for the temperature at $t = \pi/2 + 0.01$.

24. Let $f(x,y) = xy + (x + 3y)^2$. Find f_{xx}, f_{yy}, f_{xy}, and f_{yx}.

25. Let $g(x, y) = \sin x \sin^2 y$. Find all four second-order partial derivatives.

26. Let $u(x, y, z) = \exp(xy) + z \cos x$. Find all nine second-order partial derivatives.

27. Let $g(x, y) = 2xy / (x^2 + y^2)^2$. Find all four second-order partial derivatives.

28. Let $w(x, y) = \exp(-xy^2) + y^3x^4$. Find all four second-order partial derivatives.

29. (*Thomas*) The lengths *a, b, c* of the edges of a rectangular box are changing with time. At the instant in question, a = 1 in, b = 3 in, c = 4 in, da/dt = db/dt = 3 in/sec, dc/dt = -2 in/sec. At what rates are the box's volume *V* and surface area *S* changing at that instant? Are the box's main diagonals increasing or decreasing in length?

30. (*Thomas*) Suppose that the partial derivatives of a function f(x, y, z) at points on the helix x = cos t, y = sin t, z = t are given by

$$f_x = \cos t,\ f_y = \sin t,\ f_z = t^2 + t - 2.$$

At which points on the helix, if any, can f assume extreme values?

31. (*Thomas*) Let T = f(x,y) be the temperature at the point (x, y) on the circle x = cos t, y = sin t, $0 \le t \le 2\pi$ and suppose that $T_x = 8x - 4y$, $T_y = 8y - 4x$.

(a) Find the maximum and minimum temperatures on the circle by examining dT/dt and d^2T/dt^2.

(b) Given that $T = 4x^2 - 4xy + 4y^2$, find the max and min values of T on the circle.