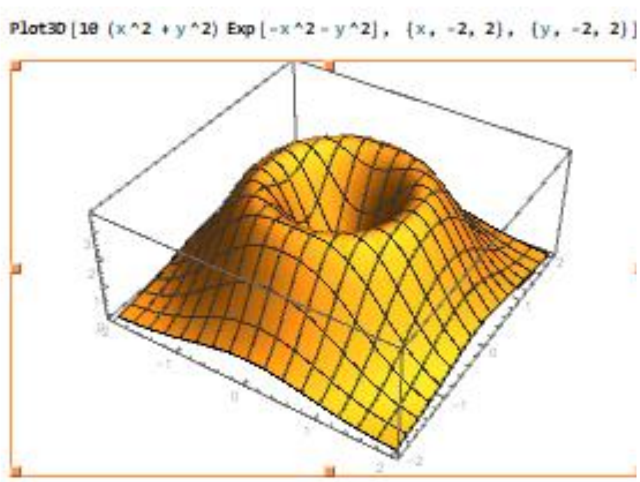
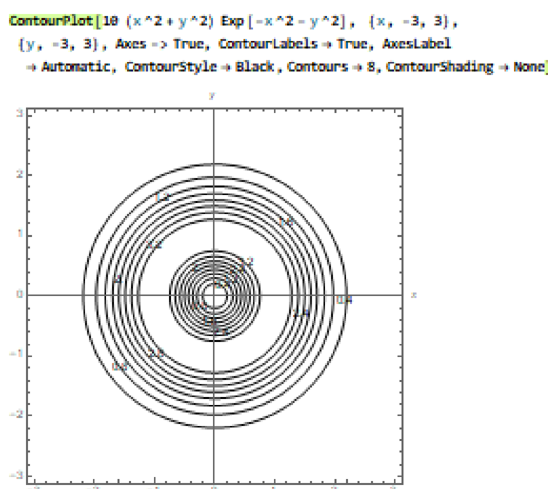
MATH 263 CLASS DISCUSSION 1 MARCH 2019

GLOBAL EXTREMA, COMPACTNESS

1. Review: Show that the critical points of the "volcano"

$y = 10(x^2 + y^2)e^{-(x^2+y^2)}$ occur at (0, 0) and on the circle $x^2 + y^2 = 1$.



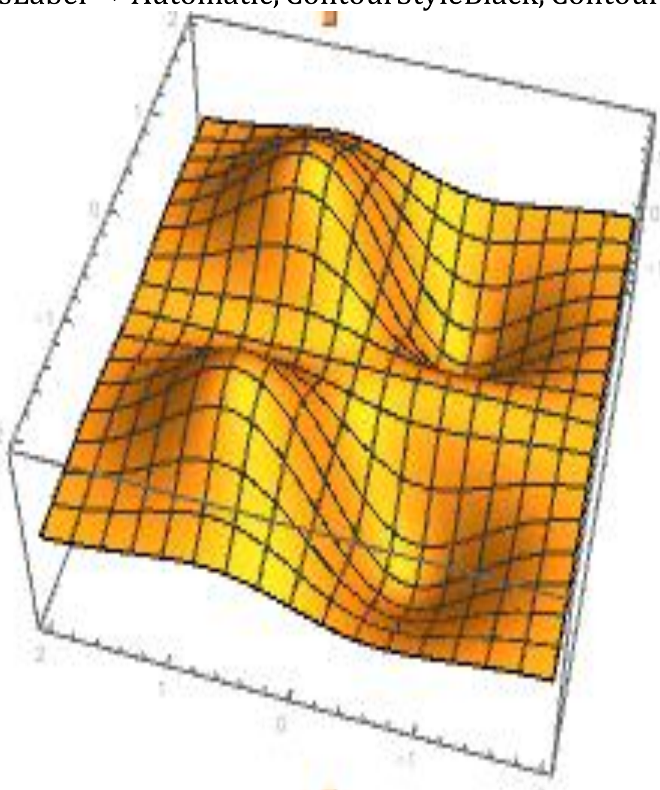


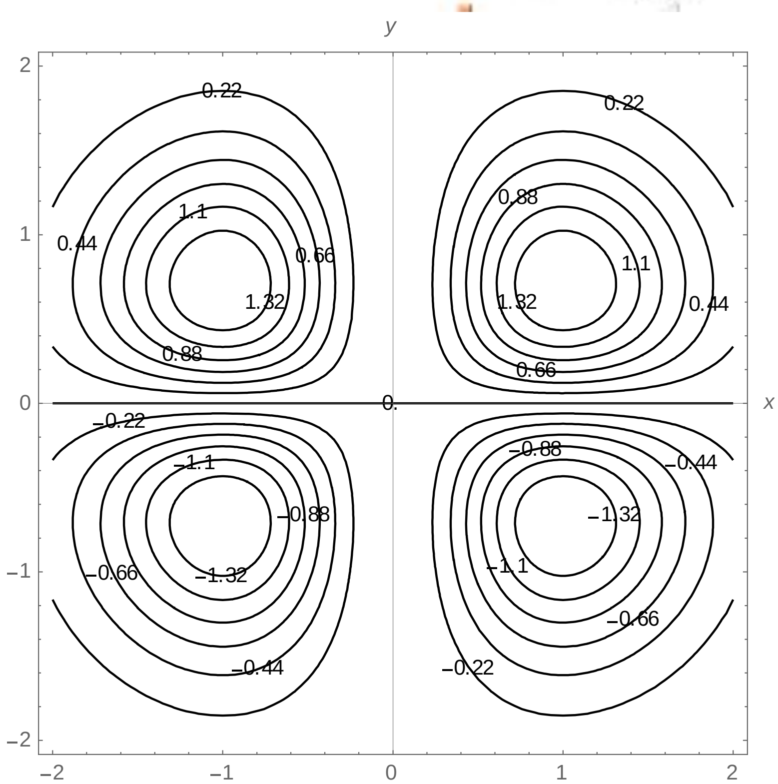
2. Review: Show that $y = 10\, x^2 y\, e^{-(x^2+y^2)}$ has maximum values at $\left(\pm 1, \frac{1}{\sqrt{2}}\right)$ and minimum values at $\left(\pm 1, -\frac{1}{\sqrt{2}}\right)$. Show also that *f* has infinitely many other critical points and D = 0 at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

3.

Plot3D[10x^2yExp[−x^2 − y^2], {x, −2,2}, {y, −2,2}]

ContourPlot[10x^2yExp[−x^2 − y^2], {x, −2,2}, {y, −2,2}, Axes−> True, ContourLabels → True, AxesLabel → Automatic, ContourStyleBlack, ContoursContourShading → None]





Compact Domains

4. What does it mean for a subset of $\mathbf{R}^2$ (or of $\mathbf{R}^3$) to be open? closed? bounded? compact? What is meant by the boundary of a set S? This will be represented by the symbol ∂S.

5. State the Compactness Theorem for continuous functions.

6. Find the global max and global min of $f(x, y) = x^2 + xy + y^2$ over the square $[-1, 1] \times [1, 1]$.

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ContourPlot[x^2 + x y + y^2, {x, -1, 1}, {y, -1, 1}, Axes -> True, ContourLabels → True, AxesLabel
  → Automatic, ContourStyle → Black, Contours → 7, ContourShading → None]
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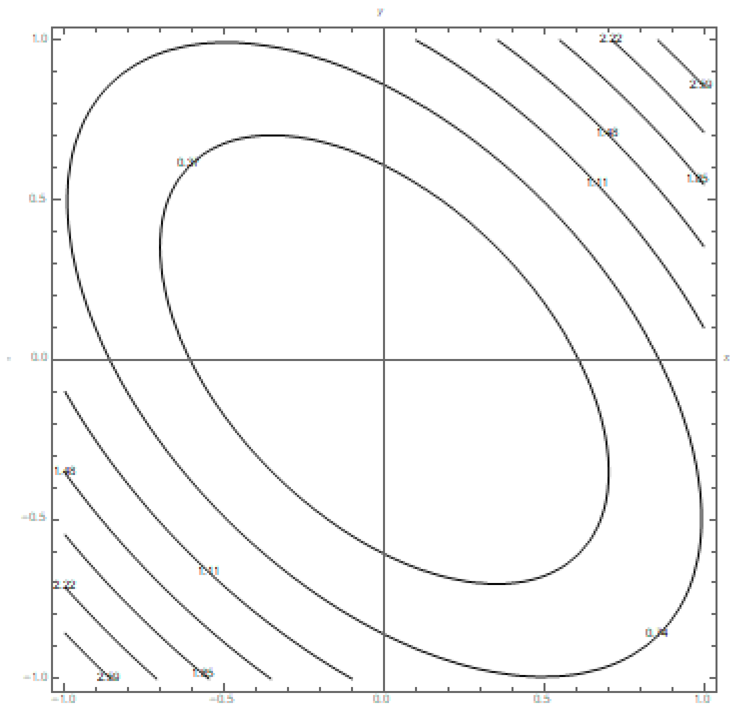

7. Find three positive numbers whose sum is 100 and whose product is maximum.

8. In each of the following exercises (31 – 38) from Stewart, find the global max and min values of f on the compact set D.

31. $f(x, y) = x^2 + y^2 - 2x$, D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$

Answer

32. $f(x, y) = x + y - xy$, D is the closed triangular region with vertices $(0, 0)$, $(0, 2)$, and $(4, 0)$

33. $f(x, y) = x^2 + y^2 + x^2y + 4, \quad D = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 1\}$

35. $f(x, y) = x^2 + 2y^2 - 2x - 4y + 1$, $D = \{(x, y) \mid 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 3\}$

36. $f(x, y) = xy^2$, $D = \{(x, y) \mid x \geqslant 0, y \geqslant 0, x^2 + y^2 \leqslant 3\}$

37. $f(x, y) = 2x^3 + y^4$, $D = \{(x, y) \mid x^2 + y^2 \leqslant 1\}$

Answer

38. $f(x, y) = x^3 - 3x - y^3 + 12y$, D is the quadrilateral whose vertices are $(-2, 3)$, $(2, 3)$, $(2, 2)$, and $(-2, -2)$

9.

A rectangular building is being designed to minimize heat loss. The east and west walls lose heat at a rate of 10 units/m^2 per day, the north and south walls at a rate of 8 units/m^2 per day, the floor at a rate of 1 unit/m^2 per day, and the roof at a rate of 5 units/m^2 per day. Each wall must be at least 30 m long, the height must be at least 4 m, and the volume must be exactly 4000 m^3.

a. Find and sketch the domain of the heat loss as a function of the lengths of the sides.

b. Find the dimensions that minimize heat loss. (Check both the critical points and the points on the boundary of the domain.)

c. Could you design a building with even less heat loss if the restrictions on the lengths of the walls were removed?

10. If the length of the main diagonal of a rectangular box must be L, what is the largest possible volume of the box?

11. By parameterizing the boundary and using the second derivative test on the interior of the given domain, find the global extrema (if such exist) of:

(a) (S. Colley, **Vector Calculus**) Let $f(x, y) = x^2 - xy + y^2 + 1$ on the closed square, S, given by $[-1, 2] \times [-1, 2]$. (*Hint*: There will be one critical point in the interior of S and 8 critical points of f restricted to the boundary of S.

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Plot3D[x^2 - x y + y^2 + 1, {x, -1, 2}, {y, -1, 2}]
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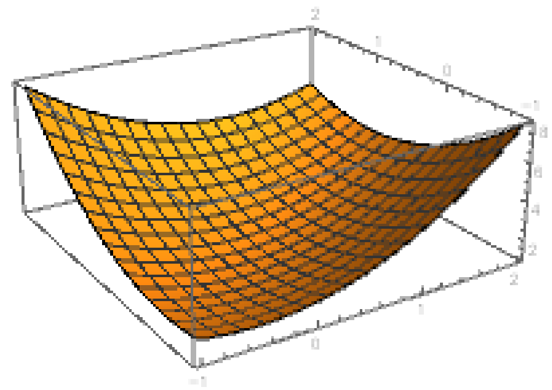

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ContourPlot[x^2 - x y + y^2 + 1, {x, -1, 2}, {y, -1, 2}, Axes -> True, ContourLabels → True, AxesLabel
  → Automatic, ContourStyle → Black, Contours → 11, ContourShading → None]
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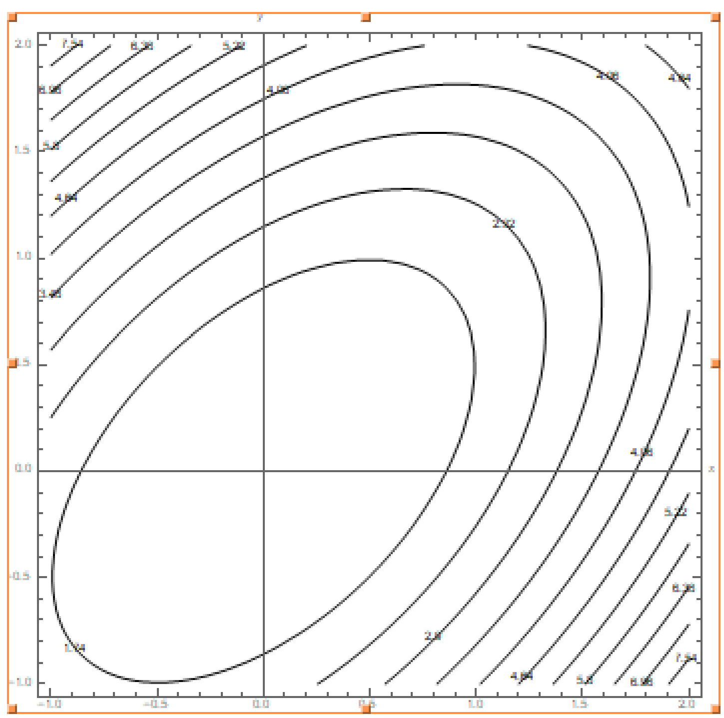

(b) $F(x, y) = 2x^2 + y^2 - 4x - 2y + 3$ on the rectangle R defined by $0 \le x \le 3$, $0 \le y \le 2$.

(c) $H(x, y) = y - x^2$ on the region whose boundary is a triangle with vertices (0, 0), (2, 0), (0, 2).

(d) $f(x, y) = x^2 + y^2 - x - y + 1$ on the disc $x^2 + y^2 \le 1$.

(e) $g(x, y) = \sin x + \cos y$ on the rectangle R defined by $0 \le x \le 2\pi$, $0 \le y \le 2\pi$.

(f) $F(x, y) = xy$ on the rectangle R defined by $-1 \le x \le 1$, $-1 \le y \le 1$.

(g) $G(x, y) = x^2 + 4y^2$ on the disc $x^2 + y^2 \le 1$.