CLASS DISCUSSION: LAGRANGE MULTIPLIERS & GLOBAL EXTREMA

11 MARCH 2019

I Review: By parameterizing the boundary and using the second derivative test on the interior of the given domain, find the global extrema (if such exist) of:

(a) (S. Colley, Vector Calculus) Let $f(x, y) = x^2 - xy + y^2 + 1$ on the closed square, *S*, given by $[-1, 2] \times [-1, 2]$. (Hint: There will be one critical point in the interior of *S* and 8 critical points of *f* restricted to the boundary of *S*.

(b) $F(x, y) = 2x^2 + y^2 - 4x - 2y + 3$ on the rectangle *R* defined by $0 \le x \le 3$, $0 \le y \le 2$.

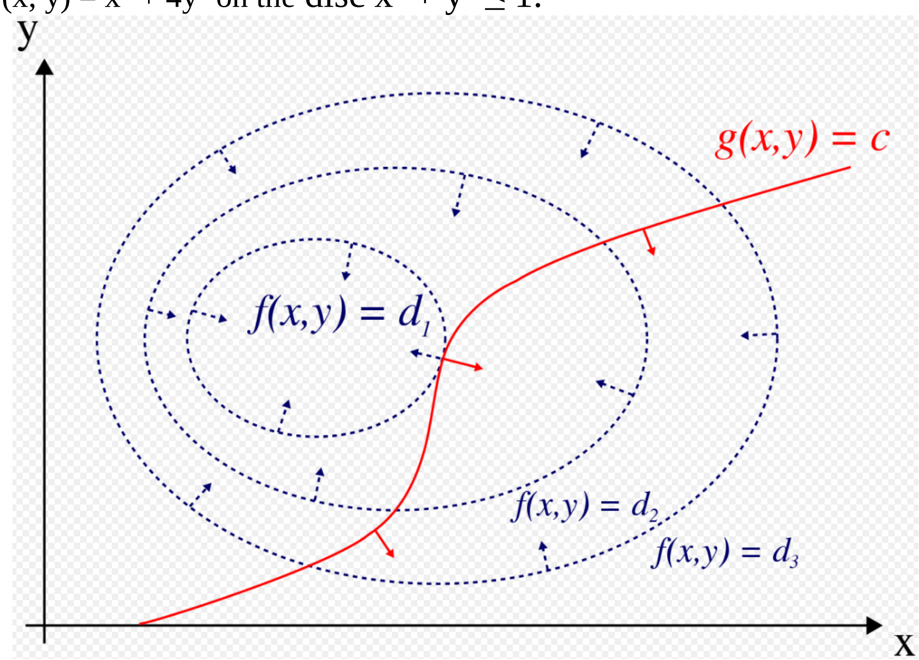
(c) $H(x, y) = y - x^2$ on the region whose boundary is a triangle with vertices (0, 0), (2, 0), (0, 2).

(d) $f(x, y) = x^2 + y^2 - x - y + 1$ on the disc $x^2 + y^2 \le 1$.

(e) $g(x, y) = \sin x + \cos y$ on the rectangle *R* defined by $0 \le x \le 2\pi$, $0 \le y \le 2\pi$.

(f) $F(x, y) = xy$ on the rectangle *R* defined by $-1 \le x \le 1$, $-1 \le y \le 1$.

(g) $G(x, y) = x^2 + 4y^2$ on the disc $x^2 + y^2 \le 1$.



II Using the method of **Lagrange multipliers:**

(a) Find all global extrema of the function $f(x, y) = x^2 - 2y$ subject to the constraint that $x^2 + y^2 = 9$.

(b) Find all local extrema of the function $f(x, y) = x^2 + y^2$ subject to the constraint that

$$y - x - 1 = 0.$$

(c) Let $f(x, y) = 2x^2 + y^2 - 2y - 4$ subject to the constraint that $x^2 + y^2 \leq 4$.

(d) Find the global extrema of $f(x, y) = 3x + 2y$ subject to the constraint

$$2x^2 + 3y^2 = 3.$$

(e) Find all global and local extrema of $f(x, y) = x^2 + y^2$ subject to the constraint that

$$x^2 - y^2 = 1.$$

(f) Find all global and local extrema of $f(x, y) = x^3 + y$ subject to the constraint that

$$x + y \geq 1.$$

(g) Find all global and local extrema of $f(x, y) = x^2 + y^2$ subject to the constraint that

$$x^4 + y^4 = 2.$$

(h) Find all global and local extrema of $f(x, y) = x^3 - y^3$ subject to the constraint that

$$x^2 + y^2 \leq 1.$$

(i) Consider the function $z = y^2 - x^2$. On the xy-plane, draw the constraint triangle having vertices (0,0), (1,2), and (2,-2). Determine the points at which *z* achieves a global maximum and global minimum.

(j) Determine the location of the peak(s) of the mountain defined by

$$z = x^2 + xy + y^2 - 6x + 20 \text{ on the domain } [0, 5]\times[-3, 3].$$

(k) A triangular plate has vertices (0, 4), (0, -2), and (3, -2). The temperature at each point of the plane is given by $T(x, y) = x^2 + xy + 2y^2 - 3x + 2y$. Determine the hottest and coldest points of the plate.

(l) Find all global extrema of the function $w = x + 2y + z$ subject to the constraint that $x^2 + 4y^2 = z$.

(m) Find all global extrema of the function $w = x + y + z$ subject to the constraint that $x^2 + y^2 + z^2 = 4$.

(n) Find all global extrema of the function $w = x^2 + y^2 + z^2$ subject to the constraint that $y - x = 1$.

III (a) Find the point on the plane $2x - y + 2z = 20$ nearest the origin. (*Hint:* Minimize the square of the distance.)

(b) Show that a rectangular box of given volume has minimum surface area when the box is a cube.

(c) Postal regulations specify that the combined length and girth of a package sent by priority mail may not exceed 108 inches. Find the dimensions of a rectangular package with maximum volume satisfying the postal regulations. (*Note:* For a rectangular box, the *girth* is defined to be 2(height + width), i.e. the perimeter of a cross section perpendicular to its length.)

(d) Find the point(s) on the surface $z^2 - xy = 1$ that are closest to the origin.

(e) Find the dimensions of an open rectangular box of maximum volume that can be constructed from 48 ft^2 of cardboard.

(f) Find the dimensions of a closed rectangular box of maximum volume that can be constructed from 48 ft^2 of cardboard.

IV **Lagrange multipliers in 3 independent variables** subject to one constraint.

(a) Find the minimum distance from the origin to the surface $xy + 2xz = 5^{3/2}$.

(b) Albertine plans to vacation in Rome, Paris and Nice for *x, y* and *z* days, respectively. Through self-reflection she determines that her enjoyment for such a vacation is given by $f(x,y,z) = x + 3x + 4y$. She also realizes that her financial constraints require that $x^2 + y^2 + z^2 = 225$. Determine how best to maximize Albertine's enjoyment.

V (Optional, if time permits) Using Lagrange multipliers in the case of two constraints.

(a) Maximize the function $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x - y = 0$ and $y + z = 0$.

(b) Find the global extrema of $f(x, y, z) = xy + z^2$ on the circle in which the plane $y - x = 0$ intersects the sphere $x^2 + y^2 + z^2 = 4$.