CLASS DISCUSSION: 20 MARCH 2019

DOUBLE INTEGRALS IN POLAR COORDINATES

VERNAL EQUINOX

Review:

1. Using a double integral, find the area of an ellipse with semi-axes of length a and b.

2. Compute the *volume* of each of the following regions:

(a) Beneath the saddle z = xy and above the unit square [0, 1]×[0, 1]
(b) Beneath the paraboloid $z = x^2 + y^2$ and above the rectangle [-1, 1]×[0, 1]

3. Compute the *average temperature* over the given plate:

(a) T(x, y) = x sin y; R = region enclosed by the curves y = 0, $y = x^2$, and x = 1
(b) $T(x, y) = y/(1 + x^2)$; R = [0, π]×[0, 2π]
(c) T(x, y) = y ln x – 5; R = [1, e]×[1/e, 1]

4. Find the *mass* of each of the following metallic sheets given the density function:

(a) $\delta(x, y) = x\, e^{x+y}$ g/cm^2; R = [0, 2]×[0, 9]
(b) $\delta(x, y) = x \sin^2 y$ lb/ft^2; R = [1, 3]×[1, 9]

5. Evaluate each of the following double integrals over the given region. Begin by sketching the region of integration. Identify whether the region is x-simple, y-simple, or neither.

$$(a)\quad \iint_R \sqrt{1-x^2}\,dA,\quad R=\{(x,y): 0\le x\le 1,\ 0\le y\le x\}$$

$$(b)\quad \iint_R (x^3+2y)\,dA,\quad R=\{(x,y): 0\le x\le 2,\ x^2\le y\le 2x\}$$

$$(c)\quad \iint_R xy\,dA,\quad R=\{(x,y): -1\le x\le 2,\ -x^2\le y\le 1+x^2\}$$

$(d)\ \iint_R (1+2x+2y)\,dA,\ \ R=\{(x,y): 0\le y\le 1,\ y\le x\le 2y\}$

6. Evaluate each of the following iterated integrals by *reversing the order* of integration. (The first step is to identify the region of integration!)

$(a)\ \int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy \qquad (b)\ \int_0^1\int_{2y}^2 e^{-x^2}\,dx\,dy \qquad (c)\ \int_0^4\int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx \qquad (d)\ \int_0^4\int_{\sqrt{y}}^2 \frac{1}{\sqrt{x^3+1}}\,dx\,dy$

Hughes-Hallett problems:

In Exercises 1–4, sketch the region of integration.

1. $\int_0^{\pi}\int_0^{x} y\sin x\,dy\,dx$ **2.** $\int_0^1\int_{y^2}^{y} xy\,dx\,dy$

3. $\int_0^2\int_0^{y^2} y^2x\,dx\,dy$ **4.** $\int_0^1\int_{x-2}^{\cos\pi x} y\,dy\,dx$

For Exercises 5–12, evaluate the integral.

5. $\int_0^3\int_0^4 (4x+3y)\,dx\,dy$ **6.** $\int_0^2\int_0^3 (x^2+y^2)\,dy\,dx$

7. $\int_0^3\int_0^2 6xy\,dy\,dx$ **8.** $\int_0^1\int_0^2 x^2y\,dy\,dx$

9. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy$ **10.** $\int_0^2\int_0^y y\,dx\,dy$

11. $\int_0^3\int_0^y \sin x\,dx\,dy$ **12.** $\int_0^{\pi/2}\int_0^{\sin x} x\,dy\,dx$

For Exercises 13–16, sketch the region of integration and evaluate the integral.

13. $\int_1^3\int_0^4 e^{x+y}\,dy\,dx$ **14.** $\int_0^2\int_0^x e^{x^2}\,dy\,dx$

15. $\int_1^5\int_x^{2x} \sin x\,dy\,dx$ **16.** $\int_1^4\int_{\sqrt{y}}^{y} x^2y^3\,dx\,dy$

Challenge Problems [Caltech]

A.

(10 Points)**Section 5.4, Exercise 2(a).** *Find*

$$\int_{-1}^{1}\int_{|y|}^{1}(x+y)^2dx\,dy.$$

B.

(10 Points)**Section 5.3, Exercise 2(a).** *Evaluate and sketch the region of integration*

$$\int_{-3}^{2}\int_{0}^{y^2}(x^2+y)dx\,dy.$$

C.

(10 Points)**Section 5.4, Exercise 8.** *Compute the double integral*

$$\iint_D f(x,y)dA$$

where

$$f(x,y)=y^2\sqrt{x}$$

and D is the set of (x,y) where $x>0, y>x^2$, and $y<10-x^2$.

POLAR COORDINATES:

I Convert each of the following double integrals into a double integral in polar coordinates and evaluate. In each case, sketch the region of integration in the xy-plane as well as in the $r\theta$-plane.

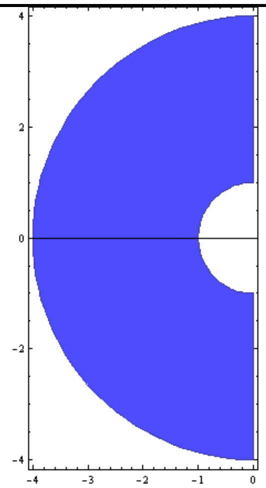
I.

$(a)\ \int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x\,dy\,dx$ $\qquad (b)\ \int_{0}^{\sqrt{5}}\int_{-x}^{x} 1\,dy\,dx$ $\qquad (c)\ \int_{0}^{\sqrt{2}}\int_{y}^{\sqrt{4-y^2}} x\,dx\,dy$

II Compute the integral of $f(x,y)=\dfrac{1}{\left(x^2+y^2\right)^{3/2}}$ over the region R of the plane defined by $1 \le R \le 2$ and $0 \le \theta \le \pi/4$. Sketch the region.

III Alphaville is located on the coast and surrounds a bay as shown below:

y(km)



ocean

x (km)

bay

The population density of Alphaville (in thousands of people per square km) is $\delta(r, \theta)$ where r and θ are the polar coordinates and distance, measured in km.

(a) Write an iterated integral in polar coordinates that expresses the total population of Alphaville.

(b) The population density decreases the farther you live from the shoreline of the bay; it also decreases the farther you live from the ocean. Which of the functions below best describes this situation:

- (i) $\delta(r, \theta) = (4 - r)(2 + \cos\theta)$
- (ii) $\delta(r, \theta) = (4 - r)(2 + \sin\theta)$
- (iii) $\delta(r, \theta) = (4 + r)(2 + \cos\theta)$

(c) Using your choice of density from part (b), calculate the population of Alphaville.

IV An ice-cream cone can be modeled by the region bounded by the hemisphere

$$z=\sqrt{8-x^2-y^2}$$

and the cone

$$z=\sqrt{x^2+y^2}$$

Distance is measured in inches. Find the volume of the "ice-cream cone".

V Let R be the unit disk, centered at the origin. Calculate $\iint_R \sin\left(x^2+y^2\right)dA$

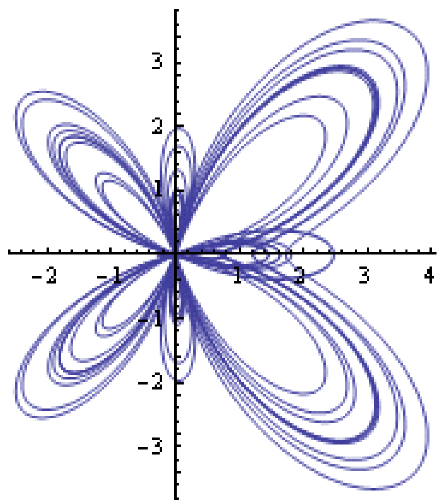
VI A disk of radius 7 cm has density 11 gm/cm^2 at its center, density 0 at its edge, and its density is a linear function of the distance from the center. Find the *mass* of the disk.

VII Consider the integral $\int_0^3 \int_{x/3}^1 f(x,y)\, dy\, dx$.

(a) Sketch the region over which the integration is defined.

(b) Rewrite the integral with the order of integration reversed.

(c) Rewrite the integral in polar coordinates.



PolarPlot[Exp[Cos[θ]] – 2 Cos[4 θ] + Sin[θ /12]^5, {θ, 0, 2π}]