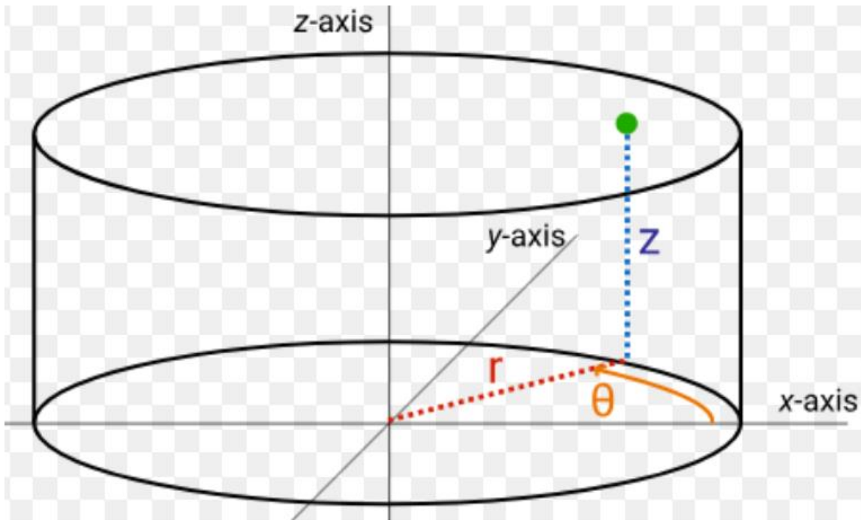
CLASS DISCUSSION 27 MARCH 2019

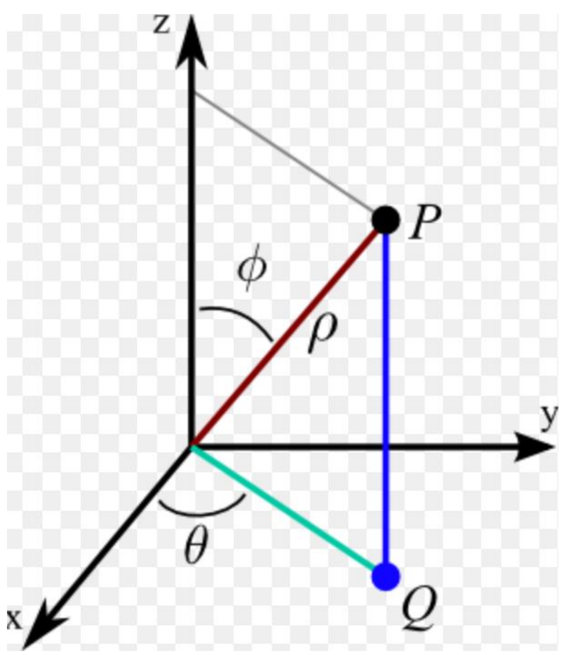
CYLINDRICAL & SPHERICAL CHANGE OF VARIABLES

I Coordinate Systems

Cylindrical Coordinates



Spherical Coordinates



1. (a) Find and plot the *cylindrical coordinates* of (6, 6, 8).

(b) If a point has *cylindrical coordinates* (8, $2\pi/3$, -3), what are its *Cartesian coordinates*?

2. Describe the surfaces r = constant, θ = constant, and z = constant in the *cylindrical coordinate system*.

3. The following points are in *cylindrical coordinates*; express each in terms of *rectangular coordinates*: (1, 45°, 1), (2, $\pi/2$, -4), (0, 45°, 10), (3, $\pi/6$, 4), (1/ $\pi/6$, 0), (2, $3\pi/4$, -2).

4. (a) Find the *spherical coordinates* of the Cartesian point (1, -1, 1).

(b) Find the *Cartesian coordinates* of the spherical coordinate point (3, $\pi/6$, $\pi/4$).

(c) Find the *spherical coordinates* of the point having rectangular

coordinates (2, -3, 6).

(d) Find the *Cartesian coordinates* of the point having spherical coordinates

(1, $-\pi/2$, $\pi/4$).

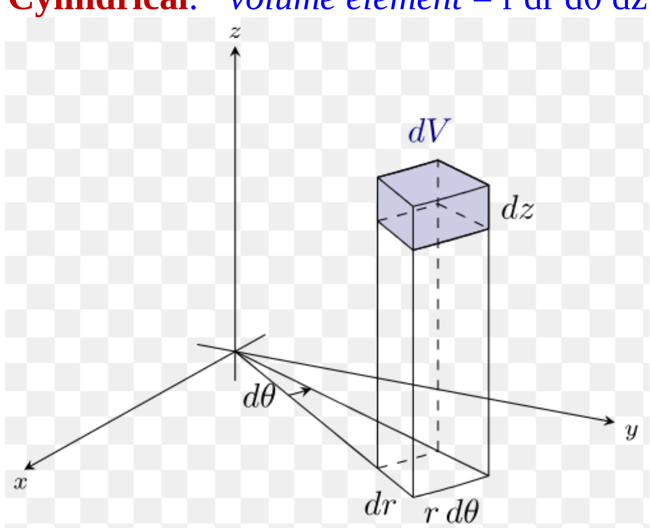
5. Describe the surfaces ρ = constant, θ = constant, and ϕ = constant in the spherical coordinate system.

6. Change each of the following points from *rectangular coordinates* to *spherical coordinates* and to *cylindrical coordinates*:

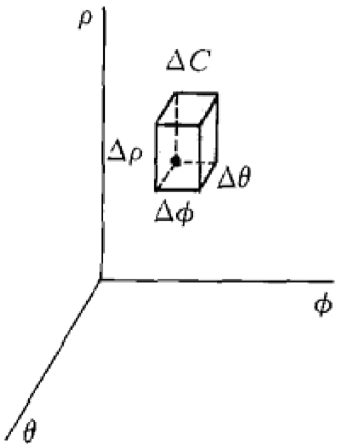
(2, 1, -2), (0, 3, 4), ($\sqrt{2}$, 1, 1), ($-2\sqrt{3}$, -2, 3)

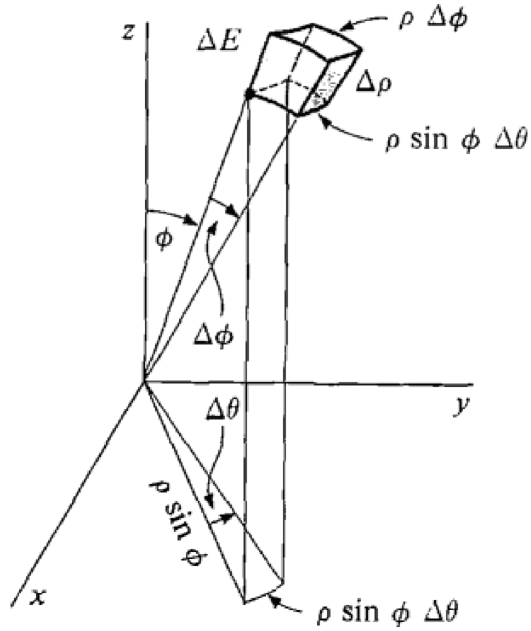
II triple integrals in cylindrical and spherical coordinates

Cylindrical: *volume element* = r dr dθ dz



Spherical: *volume element* = $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$





1. A wedge of cheese in the first octant is bounded by the cylinder $x^2 + y^2 = 4$, the xy-plane, the plane $z = 3$, the xz-plane and the plane $x = y$. Suppose that the density of the cheese is non-uniform and given by $\delta(x, y, z) = 2x + 2y + z$ ounces/in^3. Find the total weight of the cheese.

2. Evaluate the following integral where *B* is the solid region inside the cylinder $x^2 + y^2 = 1$ above the xy-plane and below the cone $z = (x^2 + y^2)^{1/2}$.

$$\iiint_{\mathbf{B}} z \, dx \, dy \, dz$$

3. Express as a triple integral using cylindrical coordinates the volume of a region *R* that is bounded below by the plane $z = 0$, on the side by the cylinder $r = \cos\theta$, and on top by the paraboloid $z = 3r^2$.

4. Convert the following integral from Cartesian coordinates to cylindrical coordinates:

$$\int_{-1}^{1} \int_{0}^{\sqrt{1-y^2}} \int_{0}^{x} (x^2 + y^2) \; dz \, dx \, dy$$

5. Using spherical coordinates, derive the formula for the volume of a ball of radius *r* centered at (0, 0, 0).

6. Suppose that the density of the unit ball at (x, y, z) is given by $\delta(x, y, z) = \exp((x^2 + y^2 + z^2)^{3/2})$ gm/cm^3. Using spherical coordinates, express the *total mass* of the ball as a triple integral, and evaluate it.

7. Evaluate the following triple integral where *W* is the solid bounded by the two spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

$$\iiint_W \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$$

8. Suppose that the temperature (in degrees Centigrade) at a point (x, y, z) in the region W consisting of a sphere centered at the origin of radius 4.

$$T(x, y, z) = (x^2 + y^2 + z^2)^{1/2} \exp(-(x^2 + y^2 + z^2)).$$

Find the *average temperature* of the solid *W.*

III Stewart Problems:

1 and 2 Plot the point whose cylindrical coordinates are given. Then find the rectangular coordinates of the point.

1.

a. $(4, \pi/3, -2)$

Answer

b. $(2, -\pi/2, 1)$

Answer

2.

a. $(\sqrt{2}, 3\pi/4, 2)$

b. $(1, 1, 1)$

9 and 10 Write the equations in cylindrical coordinates.

9.

a. $x^2 - x + y^2 + z^2 = 1$

Answer

b. $z = x^2 - y^2$

Answer

10.

a. $2x^2 + 2y^2 - z^2 = 4$

b. $2x - y + z = 1$

13. A cylindrical shell is 20 cm long, with inner radius 6 cm and outer radius 7 cm. Write inequalities that describe the shell in an appropriate coordinate system. Explain how you have positioned the coordinate system with respect to the shell.

Answer

14. Use a graphing device to draw the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 5 - x^2 - y^2$.

15 and 16 Sketch the solid whose volume is given by the integral and evaluate the integral.

15. $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^2 \int_0^{r^2} r \, dz \, dr \, d\theta$

Answer

16. $\displaystyle\int_0^2 \int_0^{2\pi} \int_0^r r \, dz \, d\theta \, dr$

17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27 and 28 Use cylindrical coordinates.

17. Evaluate $\displaystyle\iiint_E \sqrt{x^2 + y^2} \, dV$, where E is the region that lies inside the cylinder $x^2 + y^2 = 16$ and between the planes $z = -5$ and $z = 4$.

Answer

18. Evaluate $\displaystyle\iiint_E z \, dV$, where E is enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

19. Evaluate $\displaystyle\iiint_E (x + y + z) \, dV$, where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$.

Answer

20. Evaluate $\displaystyle\iiint_E (x - y) \, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$, above the xy-plane, and below the plane $z = y + 4$.

21. Evaluate $\displaystyle\iiint_E x^2 \, dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

Answer

22. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

9 and 10 Write the equation in spherical coordinates.

9.

a. $x^2 + y^2 + z^2 = 9$

Answer

b. $x^2 - y^2 - z^2 = 1$

Answer

10.

a. $z = x^2 + y^2$

b. $z = x^2 - y^2$

11, 12, 13 and 14 Sketch the solid described by the given inequalities.

11. $\rho \leqslant 1, \quad 0 \leqslant \phi \leqslant \pi/6, \quad 0 \leqslant \theta \leqslant \pi$

Answer

12. $1 \leqslant \rho \leqslant 2, \quad \pi/2 \leqslant \phi \leqslant \pi$

13. $2 \leqslant \rho \leqslant 4, \quad 0 \leqslant \phi \leqslant \pi/3, \quad 0 \leqslant \theta \leqslant \pi$

Answer

14. $\rho \leqslant 2, \quad \rho \leqslant \csc \phi$

15. A solid lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. Write a description of the solid in terms of inequalities involving spherical coordinates.

Answer

16.

a. Find inequalities that describe a hollow ball with diameter 30 cm and thickness 0.5 cm. Explain how you have positioned the coordinate system that you have chosen.

b. Suppose the ball is cut in half. Write inequalities that describe one of the halves.

17 and 18 Sketch the solid whose volume is given by the integral and evaluate the integral.

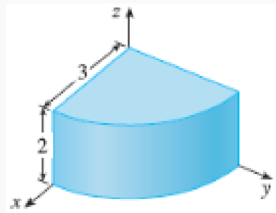
17. $\displaystyle\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

Answer

18. $\displaystyle\int_0^{\pi/4} \int_0^{2\pi} \int_0^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

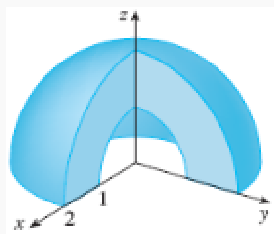
19 and 20 Set up the triple integral of an arbitrary continuous function $f(x, y, z)$ in cylindrical or spherical coordinates over the solid shown.

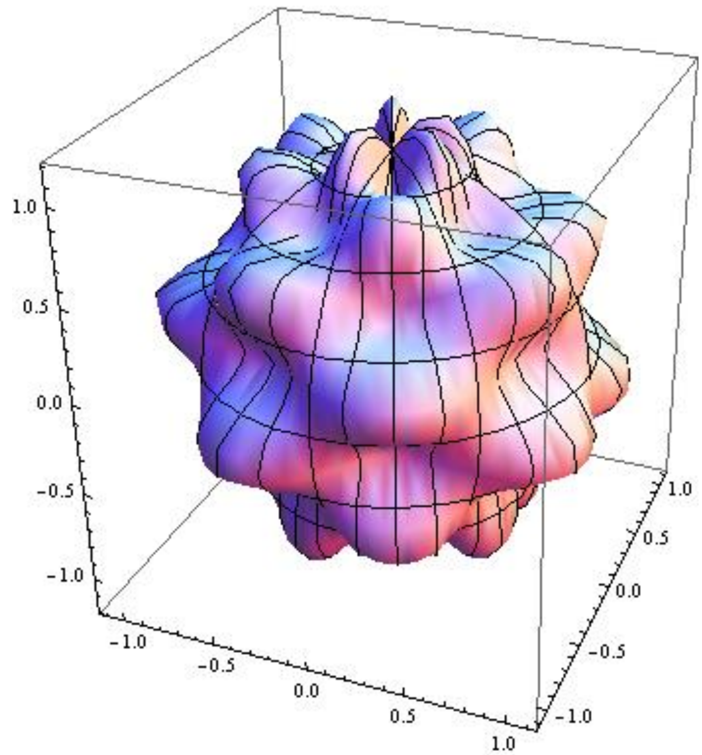
19.



Answer

20.





SphericalPlot3D[$1 + 0.2\mathrm{Sin}[8\theta]\mathrm{Sin}[4\phi], \{\theta, 0, 2\mathrm{Pi}\}, \{\phi, 0, \mathrm{Pi}\}$]

This "Bumpy Sphere" has been used to model tumors.
Cf. *Heat therapy for tumors: Applications of calculus to medicine* (UMAP modules in undergraduate mathematics and its applications), Leah Edelstein-Keshet

He is unworthy of the name of man who is ignorant that the diagonal of a square is incommensurate with its side.

- Plato, quoted in Memorabilia Mathematica, by R. E. Moritz.