# MATH 263: PRACTICE PROBLEMS FOR TEST III

Covering sections 14.8, chapter 15, 16.1, 16.2, 16.3

**1.** Let **F** and **G** be two 2-dimensional fields,

**F** = 3x **i** + 5y **j** and **G** = 3y **i** + 5x **j**.

Let $C_1$ be the circle with center (2, 2) and radius 1 oriented counterclockwise. Let $C_2$ be the path consisting of the straight-line segments from (0, 4) to (0, 1) and then from (0, 1) to (3, 1). Find each of the following line integrals. Explain your reasoning!

(a) $\int_{C_1} \boldsymbol{F} \cdot \boldsymbol{dr}$ (b) $\int_{C_2} \boldsymbol{F} \cdot \boldsymbol{dr}$ (c) $\int_{C_1} \boldsymbol{G} \cdot \boldsymbol{dr}$ (d) $\int_{C_2} \boldsymbol{G} \cdot \boldsymbol{dr}$

**2.** Find the *volume* of the region bounded by the plane ax + by + cz = 1 and the coordinate planes. (Assume that *a, b, c* are positive constants.)

**3.** If *D* is a rectangular plate defined by $1 \le x \le 2$, $0 \le y \le 1$ (measured in centimeters), and its mass density is given by $\delta(x, y) = ye^{xy}$ grams per square centimeter, integrate $\delta$ over *D* to find the mass of the plate.

**4.** Find the volume of the region bounded above by the paraboloid $z = 3x^2 + 4y^2$ and below by the square **R**: [-1, 1]×[-1, 1]

**5.** Find the volume of the region bounded above by the plane z = 2 – x – y and below by the square **S**: [-1, 1]×[-1, 1].

**6.** Find the extreme values of the function f(x, y, z) = x – y + z on the unit sphere $x^2 + y^2 + z^2 = 1$.

**7.** Find the *arc length* of the curve $\sigma(t) = (t \sin t, t, t \cos t)$ for $0 \le t \le \pi$.

**8.** You are given the following: $f(x, y) = \sin(x) + y^2$, x(u, v) = uv, and y(u, v) = u + v. Using the chain rule, compute $f_v$ when u = 0 and v = 1.

**9.** $x^2 + y^2$ind the area of each of the following surfaces.

(a) The part of the plane 5x + 3y – z + 6 that lies above the rectangle [1, $[1, 4] \times [2, 6]$.

(b) The part of the plane 6x + 4y + 2z = 1 that lies inside the cylinder $x^2 + y^2 = 25$.

(c) The part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane z = -2.

**10.** Charlotte, the spider, lives on a curve according to the rule curve $\sigma(t) = (2t, t^2, \ln t)$ defined for $t \geq 0$. Find the distance traveled by Charlotte as she travels from the point (2, 1, 0) to the point (4, 4, ln 2).

**11.** Consider the integral

$$\int_{-4}^{0} \int_{0}^{2x+8} f(x,y)\, dy\, dx + \int_{0}^{4} \int_{0}^{8-2x} f(x,y)\, dy\, dx$$

**12.** Sketch the region of integration and reverse the order of integrati

**13.** Suppose that you wish to find the global maximum and minimum values of $f(x, y) = x^2 + y^2$ subject to the constraint $x + 2y = 4$.

(a) Sketch the constraint equation and a few contour curves of *f*.

(b) On your sketch, show the location (approximately) of all local maxima and minima of *f* subject to the given constraint. How many are there?

(c) Use the method of Lagrange multipliers to find the exact location(s) of the local extrema that you found in part (b).

(d) Are the points you found in (c) local maxima or local minima? Explain.

(e) Does *f* have a global maximum or global minimum subject to the constraint? Why?

**14.** Reverse the order of integration in the following iterated integral and then evaluate:

$$\int_{0}^{1} \int_{e^y}^{e} \frac{x}{\ln x}\, dx\, dy$$

**15.** Reverse the order of integration in the following iterated integral and then evaluate:

$$\int_{0}^{6} \int_{\frac{x}{3}}^{2} x\sqrt{y^3+1}\, dy\, dx$$

**16.** Convert to polar coordinates and evaluate:

$$\int_{0}^{\sqrt{2}} \int_{y}^{\sqrt{4-y^2}} xy\, dx\, dy$$

**17.** Find the global extrema of $f(x, y) = x^2 + 2y^2 - 2x + 3$ subject to the constraint $x^2 + y^2 \le 10$. Draw the appropriate contour diagram and the constraint equation.

**18.** Does there *exist* a function $z = f(x, y)$ for which $z_x = xy^2$ and $z_y = x^3y$ ? Why?

**19.** Using Lagrange multipliers, find the maximum and minimum values achieved by the function $f(x, y) = xy$ on the ellipse $x^2/8 + y^2/2 = 1$. Sketch the constraint equation and indicate exactly where the local extrema occur.

**20.** Evaluate the following iterated integral:

$$\int_0^8 \int_{\sqrt[3]{x}}^2 \frac{1}{1+y^4}\, dy\, dx$$

*Hint:* Sketch the region of integration and then reverse the order of integration.

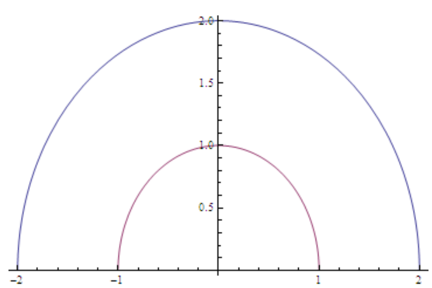
**21.** Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

**22.** Using Lagrange multipliers, find the dimensions of a rectangular box of maximum volume that can be inscribed (with edges parallel to the coordinate axes) in the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

**23.** Find the *arc length* of the curve $\sigma(t) = (\sin 3t, \cos 3t, 2t^{3/2})$ for $0 \le t \le 1$.

**24.** *Parameterize* the curve $x = 4y^3 = z^2 + 5$.

**25.** Evaluate $\iint_R \sqrt{x^2 + y^2}\; dA$ where R is the region below bounded by two circles in the upper half-plane:




**26.** Find the mass of the solid bounded by the xy-plane, yz-plane, xz-plane and the plane $x/3 + y/2 + z/6 = 1$, if the density of the solid is given by $\delta(x, y, z) = x + y$. Evaluate the integral

$$\iint_R \sin\left(x^2 + y^2\right) dA$$

where $R$ is the disk of radius 2 centered at the origin.

**27.** (a) Describe in cylindrical coordinates a wedge of cheese cut from a cylinder 4 cm high and 6 cm in radius; this wedge subtends an angle of $\pi/6$ at the center.

(b) If the density of the cheese is 1.2 grams/cm$^3$, find the mass of this wedge of cheese.

**28.** Using spherical coordinates, find the mass of the sphere of radius a > 0 centered at the origin if the density at any point is proportional to the distance of the point from the z-axis.

**29.** Notice that the following iterated triple integral cannot be evaluated since the second integration requires integration of the function sin($y^2$). First identify the region of integration and reverse the order. Finally, evaluate the integral.

$$\int_0^{\sqrt{\frac{\pi}{2}}} \int_x^{\sqrt{\frac{\pi}{2}}} \int_1^3 \sin(y^2)\, dz\, dy\, dx$$

**30.** (from Larson & Edwards) Consider a circular lawn with a radius of 10 feet. Assume that a sprinkler in the center of the disk distributes water in a radial fashion according to the formula $f(r) = r/16 - r^2/160$ (measured in cubic feet of water per hour per square foot of lawn), where r is the distance in feet from the sprinkler. Find the amount of water that is distributed in 1 hour in the following two annular regions:

(a) *A* is the region between the circle of radius 4 and the circle of radius 5 (both centered at the origin).

(b) *B* is the region between the circle of radius 9 and the circle of radius 10 (both centered at the origin).

Is the distribution of water uniform? Determine the amount of water the entire lawn receives in 1 hour.

**31.** Consider the change of variables $u = 3x - y$, $v = 3x + y$.

(a) Solve for *x* and *y* in terms of *u* and *v*.

(b) Find the *Jacobian* of this transformation (from the uv-plane into the xy-plane).

(c) Consider the region, ℝ, in the xy-plane bounded by the lines

$$3x - y = 1,\ 3x - y = 9,\ 3x + y = 6, \text{ and } 3x + y = 12.$$

Find the pre-image of ℝ in the uv-plane? *Sketch the region ℝ in the xy-plane and its pre-image in the uv-plane.*

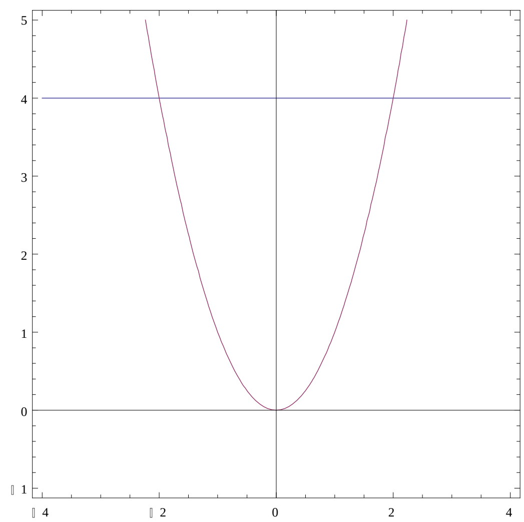
Consider the region $\mathcal{C}$ bounded above by the plane z = 8 and below by the cone $z = 4\sqrt{x^2 + y^2}$.

(a) Express the volume of $\mathcal{C}$ as a triple integral in Cartesian coordinates.

(b) Convert the integral in part (a) to one in cylindrical coordinates and evaluate.

**32.** Albertine wishes to find the global maximum and global minimum values of the function $G(x, y) = xy - x^2$ on the region ℝ bounded by the parabola $y = x^2$ and the line y = 4. She remembers that, since *G* is continuous on the closed and bounded region ℝ in the plane, the Compactness Theorem guarantees the existence of such global extrema.

(a) Albertine first locates all critical points of *G* inside ℝ. Which point(s), if any, will she find?




(b) Next, Albertine finds all local extrema on the boundary of ℝ using LaGrange multipliers. Which point(s), if any, will she find?

(c) Finally, Albertine determines where the global maximum and global minimum of *G* occur. What does she discover and what are the maximum and minimum values of *G* on ℝ?

**33.** Albertine's Bakery in BetaVille produces two types of chocolate cakes: ordinary and crafted. Each Ordinary Cake requires 0.1 lb of Swiss chocolate, while each Crafted Cake requires 0.2 lb of Swiss chocolate. Currently there are only 233 lbs of Swiss chocolate available each month. Suppose that the profit function is given by:

$$p(x, y) = 150x - 0.2x^2 + 200y - 0.1y^2$$

where *x* is the number of Ordinary cakes and *y* is the number of Crafted Cakes that the bakery produces each month.

How many of each type of chocolate cake should Albertine's bakery produce each month to *maximize profit*? *(Use LaGrange multipliers.)*

**34.** Suppose that z = f (x,y) and z = g(x,y) are smooth functions defined on the xy-plane. Prove each of the following:

(a) $\nabla(fg) = (f)(\nabla g) + (\nabla f)(g)$

(b) $\nabla (f/g) = [(g)\nabla f - (f)\nabla g] / g^2$

**35.** The two-dimensional vector field ***F*** satisfies the condition $\|\mathbf{F}(x, y)\| \le 13$ everywhere. Let *C* be is the circle of radius 4 centered at the point (5, 7). What is the largest possible value of ? $\int_C \boldsymbol{F}\cdot \boldsymbol{dr}$ ? The smallest possible value? Explain!

**36.** Let $\mathbf{F}(x, y, z) = 2xy^2z\exp(x^2y^2z)\,\vec{i} + 2x^2yz\exp(x^2y^2z)\,\vec{j} + x^2y^2\exp(x^2y^2z)\,\vec{k}$.

Compute $\int_C \vec{F}$ where *C* is the curve parameterized by $\boldsymbol{\sigma}(t) = (t, t^2, t^3)$, for $0 \le t \le 1$.

**37.** Find a *potential function* for the vector field

$$\mathbf{F}(x,y) = (3x^2 \sin y + 1 - y \sin x)\,\mathbf{i} + (x^3 \cos y + \cos x + 2y)\,\mathbf{j}.$$

**38.** (Thomas) Evaluate the integral

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

*Hint:* Make the change of variables: $u = (xy)^{1/2}$ and $v = (y/x)^{1/2}$.

*Answer: 2e(e – 2)*

**39.** (Thomas) (a) Find the Jacobian of the transformation x = u, y = uv and sketch the region G: $1 \le u \le 2$, $1 \le uv \le 2$ in the uv-plane.

(b) Using the Change of Variables Theorem, evaluate the integral

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx$$

into an integral over *G*, and evaluate both integrals.

**40.** On an exam at the Hogwart's School, students were asked to find the arc length of the curve C parameterized by

$$\sigma(t) = (\frac{t^2}{2} - \frac{t^4}{4})\vec{i} + \frac{2t^3}{3}\vec{j} + \vec{k}, \quad for\ -1 \le t \le 1.$$

Harry Potter at first wrote:

$$\|\vec{v}(t)\| = \sqrt{(t-t^3)^2 + (2t^2)^2} = \sqrt{t^2(1+2t^2+t^4)} = \sqrt{t^2(1+t^2)^2} = t(1+t^2).$$

$$Thus\ the\ arc\ length\ is\ \int_{-1}^{1} t(1+t^2)dt = 0.$$

(a) Harry realizes that his answer is wrong. Which part of Harry's calculation is incorrect?

(b) Find the correct answer to this problem.

**41.** Let $C$ be the curve from (1, -1, -1/2) to (1, 1, -1/2) parameterized by:

$$\sigma(t) = -\cos(\pi t^4)\ i + t^{\frac{5}{3}}\ j + \frac{t}{t^2+1}k \quad for\ -1 \le t \le 1$$

and let $\mathbf{F}(x,y,z) = (2xy + z^2)\,\mathbf{i} + x^2\,\mathbf{j} + (2xz + \pi\cos(\pi z))\,\mathbf{k}$. Compute $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$ (*Hint:* Find a shortcut.)

**42.** Let $C$ be the circle in space with the parameterization

$$\mathbf{r}(t) = (\cos t)\ \mathbf{i} + (\sin t)\ \mathbf{k}, \quad \text{for } 0 \le t \le 2\pi.$$

Evaluate (using *only the computational definition* of line integral)
where $\oint_C \boldsymbol{F} \cdot \boldsymbol{dr}$ where

$$\mathbf{F}(x, y, z) = z\exp(y^2)\,\mathbf{i} + \sin(x^2 + y^2 + z^2)\,\mathbf{j} + x(y^2 - 1)\,\mathbf{k}.$$

**43.** Let $P$ be the parallelogram bounded by $y = 2x$, $y = 2x - 2$, $y = x$, and $y = x + 1$. By making the change of variables $x = u - v$, $y = 2u - v$, evaluate the following integral:

$$\iint_P xy\ dA$$

**44.** Let $\mathbf{F}(x, y)$ be a conservative vector field with potential function $g(x, y)$ satisfying $g(0, 0) = 1$. Let $C_1$ be the line from (0, 0) to (2, 1), $C_2$ the path parameterized by

$$\mathbf{r}(t) = (2 + 2\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j}, \quad \text{for } 0 \le t \le \pi,$$

and $C_3$ the path parameterized by

$\boldsymbol{\sigma}(t) = (t + 2)\,\mathbf{i} + (t^2 - t + 1)\,\mathbf{j}$, for $0 \le t \le 2$.

Suppose that $\int_{C_1} F = 5,\ \int_{C_2} F = -3,\ and \int_{C_3} F = 4$ Evaluate g(2, 1), g(2, -1), and g(4, 3).

**45.** Find a potential function for the vector field:

$$\vec{F}(x,y,z) = \left(\frac{1}{x+y} + 1 + ze^{xz}\right)\vec{i} + \left(\frac{1}{x+y} + z^2\right)\vec{j} + (2yz + xe^{xz} + 13)\vec{k}$$

**46.** Find a *potential function* for the vector field

$$\vec{F}(x,y,z) = \left(y^2 - \frac{1}{x+z)}\right)\vec{i} + (2xy+5)\vec{j} + \left(3z^2 - \frac{1}{x+z}\right)\vec{k}$$

**47.** Let **G**(x, y) = $x^2\,\mathbf{i} + y^2\,\mathbf{j}$ is a vector field on the plane. Find the equation of the flow line

σ(t) that passes through the point $(\frac{1}{2}, 1)$ at time 0.

**48.** Let **F**(x, y) be a conservative vector field with potential function g(x, y) satisfying g(0, 0) = 3. Let $C_1$ be the line segment from (0, 0) to (4, 1), $C_2$ the path parameterized by **r**(t) = (4 + 2 sin (t/2)) **i** + (cos t) **j**, for $0 \le t \le \pi$, and $C_3$ the path parameterized by

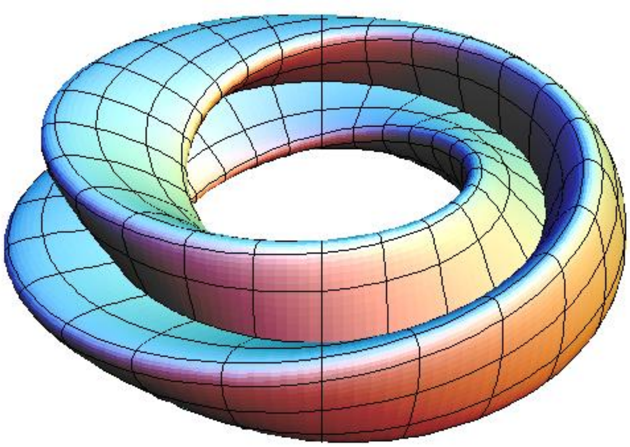
$\boldsymbol{\sigma}(t) = (t + 6)\,\mathbf{i} + (t^2 - t - 1)\,\mathbf{j}$, for $0 \le t \le 2$.

Suppose that $\int_{C_1} F = 4,\ \int_{C_2} F = -5,\ and \int_{C_3} F = 11.$

Evaluate g(4, 1), g(6, -1), and g(8, 1).

*Everyone knows what a curve is, until he has studied enough mathematics*
*to become confused through the countless number of possible exceptions.*

- Felix Klein



Klein bottle (from virtual math museum)