MATH 263 SOLUTIONS: TEST III 12 APRIL 2019

Instructions: Answer any 7 of the 9 questions. You may answer more than 7 to earn extra credit.

Volume element for cylindrical coordinates: $dV = r\,dr\,d\theta dz$

Volume element for spherical coordinates: $dV = \rho^2 \sin\varphi \;\; d\rho\, d\theta\, d\varphi$

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$x = r\cos\theta = \rho\sin\phi\,\cos\theta$

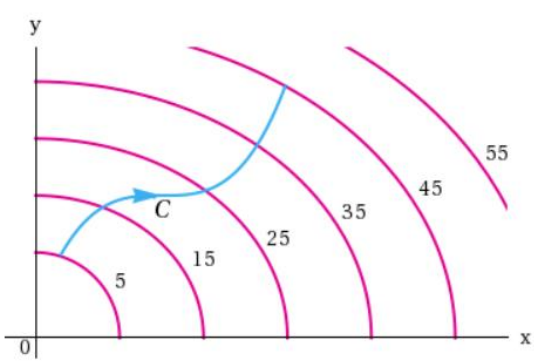
$y = r\sin\theta = \rho\sin\phi\,\sin\theta$

$z = \rho\cos\phi$

$\rho = \sqrt{r^2+z^2} = \sqrt{x^2+y^2+z^2}$

- $\rho = a$ sphere of radius a at origin
- $\phi = \frac{\pi}{4}$ cone

1. (a) The figure below shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla \boldsymbol{f} \cdot d\boldsymbol{r}$.



Solution: Since the vector field is a gradient vector field with potential function *f*, $\int_C \nabla \boldsymbol{f} \cdot \boldsymbol{dr}$ depends only upon the end points of C. Thus $\int_C \nabla \boldsymbol{f} \cdot \boldsymbol{dr} =$ $f(ending\ point\ of\ C) - f(beginning\ point\ of\ C) = 45 - 5 = 45 - 5 = 40.$

(b) Evaluate $\iiint_E \sqrt{x^2 + y^2 + z^2}\ dV$ where E is the region bounded by the two spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 25.$

Solution: Note that $x^2 + y^2 + z^2 = \rho^2$. Thus

$$\iiint_E \sqrt{x^2 + y^2 + z^2}\ dV =$$

$$\int_0^{2\pi} \int_0^{\pi} \int_2^{5} \sqrt{\rho^2}\ \ \rho^2 \sin\varphi\ \ d\rho\ d\varphi\ d\theta =$$

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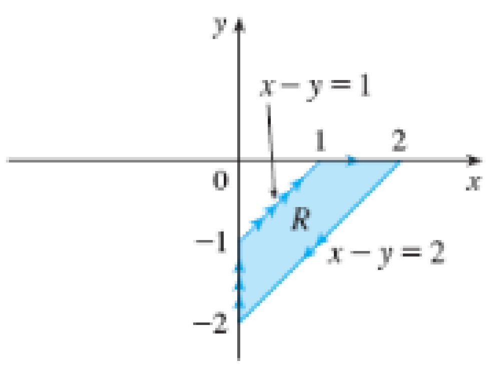
$$\int_0^{2\pi} \int_0^{\pi} \int_2^{5} \rho^3 \sin\varphi\ \ d\rho\ d\varphi\ d\theta = 609\,\pi$$

2. *[Stewart]* Let *R* be the trapezoidal region with vertices (1, 0), (2, 0), (0, -2), and (0, -1). Albertine wishes to evaluate the integral

$$\iint_R e^{\frac{x+y}{x-y}}\, dA.$$ She requests our help.

(a) Sketch the region *R* in the xy-plane.

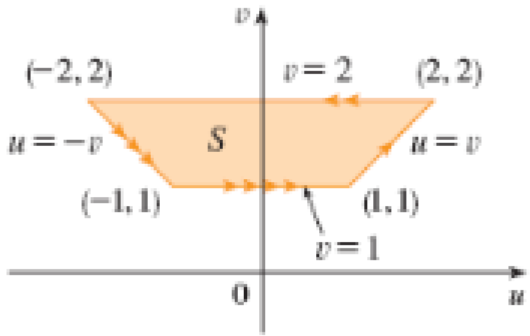
Solution:



(b) Consider the change of variables $u = x + y,\ \ v = x - y$.

Sketch the preimage of R in the uv-plane.

Solution:



Note: This region is v-simple but not u-simple

(c) Compute the Jacobian, $\frac{\partial(x,y)}{\partial(u,v)}$

Solution: Solving for x and y:

$$x = \frac{1}{2}(u+v) \qquad y = \frac{1}{2}(u-v)$$

The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

(d) Using the Change of Variables Theorem, evaluate the given integral. (Give a numerical answer.)

Solution:

$$\iint_R e^{(x+y)/(x-y)}\, dA = \iint_S e^{u/v} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

$$= \int_1^2 \int_{-v}^{v} e^{u/v} \left(\tfrac{1}{2}\right) du\, dv = \tfrac{1}{2} \int_1^2 \left[v e^{u/v} \right]_{u=-v}^{u=v} dv$$

$$= \tfrac{1}{2} \int_1^2 \left(e - e^{-1}\right) v\, dv = \tfrac{3}{4}\left(e - e^{-1}\right)$$

3. Let $\vec{F}(x,y) = (y^2 + \pi e^y \cos(\pi x))\vec{\imath} + (2xy + e^y \sin(\pi x))\vec{\jmath}$

(a) Find a *potential function* for the vector field **F**.

Solution: Begin by assuming there is a potential function, h(x). (To feel more secure, you can check to see if the curl of F is 0.)

Then $h_x = y^2 + \pi e^y \cos(\pi x)$.

Integrating with respect to x, we obtain $h(x,y) = xy^2 + e^y \sin(\pi x) + f(y)$

Next, differentiating *h* with respect to y, we obtain:

$h_y(x,y) = 2xy + e^y \sin(\pi x) + f'(y)$.

Now, we set this equal to $2xy + e^y \sin(\pi x)$

Hence $f'(y) = 0, so\ f(y)\ is\ a\ constant.$

Now we have our potential function, $\boldsymbol{h(x,y) = xy^2 + e^y \sin(\pi x)}$

(b) Let *C* be the path parametrized by $x(t) = 2\cos(\pi t),\ \ y(t) = 3 + \sin(\pi t)\ ,\ \ 0 \le t \le \frac{1}{2}$.

Compute $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$. Give a numerical answer!

Solution: Since F is a conservative field, we may use the Fundamental Theorem of Line Integrals, viz.

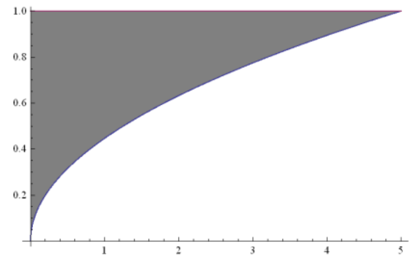
$$\int_C \boldsymbol{F} \cdot \boldsymbol{dr} = h\left(x\left(\frac{1}{2}\right), y\left(\frac{1}{2}\right)\right) - h\big(x(0).y(0)\big) = h(0,4) - 18h(2,3) = 0 - 18 = -18$$

4. For the following iterated integral, *sketch* the region of integration, reverse the order of integration, and evaluate the integral:

$$\int_0^5 \int_{\sqrt{\frac{x}{5}}}^1 e^{y^3}\, dy\, dx$$

Sketch the region. Give a numerical answer.

Solution:



This region is both x-simple and y-simple.

$$\int_0^5 \int_{\sqrt{\frac{x}{5}}}^1 e^{y^3}\,dy\,dx = \int_0^1 \int_0^{5y^2} e^{y^3}\,dx\,dy = \int_0^1 \left(xe^{y^3}\Big|_{x=0}^{x=5y^2}\right)dy$$

$$= \int_0^1 5y^2 e^{y^3}\,dy = \frac{5}{3}e^{y^3}\Big|_0^1 = \frac{5}{3}(e-1)$$

5. (a) For Halloween, Albertine is given an almost spherical pumpkin of radius 17 cm with a density function whose value (in kg/cm^3) at (x, y, z) is the *cube of the distance* to the pumpkin's center. Express in integral

form the *mass* of this pumpkin. *(Do not evaluate.)*

Solution:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{17} \rho^3\ \rho^2 \sin\phi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{17} \rho^5\ \sin\phi\,d\rho\,d\varphi\,d\theta$$

(b) Determine whether the vector field $\mathbf{F} = (y^2 - 2x)\,\mathbf{i} + 2xy\,\mathbf{j}$ is a conservative vector field. Either show why it is not conservative or else find a potential function for $\mathbf{F}$

Solution: Since curl F = 0, we expect to find a potential function.

Assume that h(x, y) is a potential function for F.

Then $h_x = y^2 - 2x$. Integrating with respect to x, we find $h(x, y) = xy^2 - x^2 + f(y)$.

Next, computing $h_y = 2xy+f'(y)$. Equating this to the j component or F, we find

$2xy = 2x = y + f'(y)$.

Thus $f'(y) - 0$, and so $f(y)$ is a consant.

Finally, $\mathbf{h(x, y) = xy^2 - x^2}$

6. (Harvard) Match the following vector fields to the pictures, below. Explain your reasoning. (Notice that in some of the pictures all of the vectors have been uniformly scaled so that the picture is more clear. Also, notice that there are eight vector fields but only six pictures. There's probably a reason behind this.)

(a) $\mathbf{F}(x, y) = \langle 1, x\rangle$

(b) $\mathbf{F}(x, y) = \langle -y, x\rangle$

(c) $\mathbf{F}(x, y) = \langle y, x\rangle$

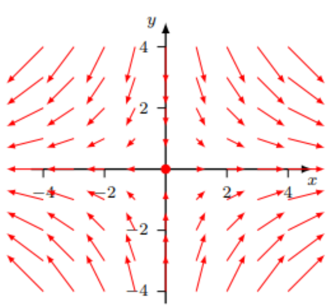
(d) $\mathbf{F}(x, y) = \langle 2x, -2y\rangle$

(e) ∇f, where $f(x, y) = x^2 + y^2$

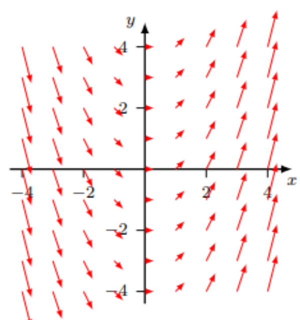
(f) ∇f, where $f(x, y) = \sqrt{x^2 + y^2}$

(g) ∇f, where $f(x, y) = xy$

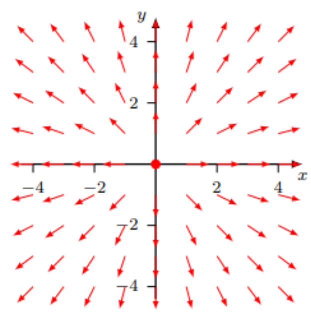
(h) ∇f, where $f(x, y) = x^2 - y^2$



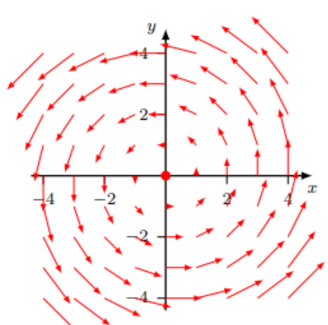
Field (I)



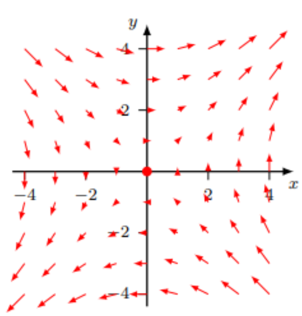
Field (II)



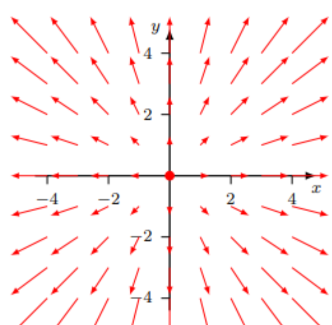
Field (III)



Field (IV)



Field (V)



Field (VI)

Answers: Field I _d and h__

Field II _a__

Field III _f__

Field IV ___b_

Field V _c and g___

Field VI _e__

7. (a) Let R be the region in the xy-plane bounded by the line y = 2x, the x-axis, and the line x = □/4. Let z = f(x, y) be defined by:

$$f(x,y) = (\cos y)e^{1-\cos(2x)} + xy$$

Express as an iterated double integral in rectangular coordinates, the *volume* of the region beneath the graph of z = *f* (x, y) and above *R*. Do not evaluate.

Solution:

$$V = \int_0^{\frac{\pi}{4}} \int_0^{2x} f(x,y)\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_0^{2x} ((\cos y)e^{1-\cos(2x)} + xy)\,dy\,dx$$

(b) Consider the following triple iterated integral expressed in cylindrical coordinates:

$$\int_0^{2\pi} \int_0^{1} \int_0^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$$

Convert this integral into an equivalent iterated integral in:

(i) Cartesian coordinates (but do not evaluate).

Solution:

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{\sqrt{9-x^2-y^2}} dz\,dx\,dy$$

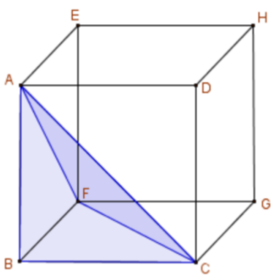
(ii) Spherical coordinates (but do not evaluate).

Solution: $\displaystyle\int_0^{2\pi}\int_0^{\arcsin\frac{1}{3}}\int_0^{3} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta + \int_0^{2\pi}\int_{\arcsin\frac{1}{3}}^{\frac{\pi}{2}}\int_0^{\csc\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

8. A solid tetrahedron with vertices (0, 0, 0), (1, 0, 0), (0, 1, 0) and (0, 0, 1) has mass density

$$\delta(x, y, z) = \sqrt{x + 2y + 3z + 13} \quad g/cm^3$$

at the point (x, y, z). Express in integral form, using Cartesian coordinates, the *mass* of this solid tetrahedron. *(Do not evaluate.)*



Solution:

The equation of the plane determined by the three points (1, 0, 0), (0, 1, 0) and (0, 0, 1), is given by z = 1 – x – y. The shadow of the solid on the xy-plane is the right triangle with vertices (0, 0), (1, 0) and (0, 1). Thus, the weight of the solid is given by the triple iterated integral

$$\int_0^1\int_0^{1-y}\int_0^{1-x-y} \delta(x, y, z)\, dz\, dx\, dy = \int_0^1\int_0^{1-y}\int_0^{1-x-y} \sqrt{x + 2y + 3z + 13}\, dz\, dx\, dy \ grams$$

9. Express as an iterated double integral the surface area of the portion of the paraboloid $z = 1 - x^2 - y^2$ that lies above the plane z = -2. Do not evaluate.

Solution: The paraboloid and the plane intersect at the circle $x^2 + y^2 = 3$ on the plane, $z = -2$.

$$A(S) = \iint_R \sqrt{1 + (z_x)^2 + \left(z_y\right)^2}\ dA = \iint_R \sqrt{1 + (-2x)^2 + (-2y)^2}\ dA =$$

$$\iint_R \sqrt{1 + (-2x)^2 + (-2y)^2}\ dA = \iint_R \sqrt{1 + 4x^2 + 4y^2}\ dA$$

Expressing this in polar coordinates:

$$\int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + 4r^2} \quad r \, dr \, d\theta$$

Water is fluid, soft and yielding. But water will wear away rock, which is rigid and cannot yield. As a rule, whatever is fluid, soft and yielding will overcome whatever is rigid and hard. This is another paradox: what is soft is strong.

- Lao-Tzu (600 B.C.)