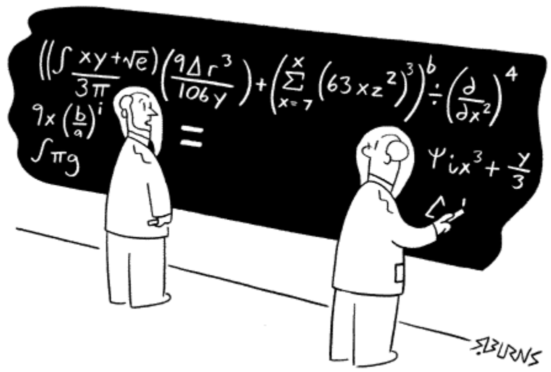
MATH 162 DISCUSSION: 29 JANUARY 2020

IMPROPER INTEGRALS



"What's the square root of infinity again?"

Reference: MIT 18.01 lecture 36 on improper integrals: https://www.youtube.com/watch?v=KhwQKE_tld0

1. Explain what is meant by an "improper integral of the **first kind**" and an "improper integral of the **second kind**." What does it mean to say that an improper integrals ***converges***? ***diverges***? ***converges to the limit L***?

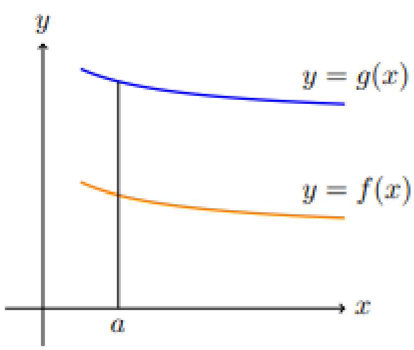
2. State and prove the p-test for improper integrals of the first kind.

3. Calculate the *exact* value of each of the following improper integrals of the first kind. (Each converges.)

$(A)\ \int_1^\infty \frac{1}{x^\pi}\,dx$ $\qquad (B)\ \int_e^\infty \frac{1}{x(\ln x)^2}\,dx$

$(C)\ \int_0^\infty te^{-t^2}dt$ $\qquad (D)\ \int_0^\infty \frac{v}{(1+v^2)^4}\,dv$

4. Discuss the comparison test for improper integrals. (Is there a difference in dealing with improper integrals of the *first kind* vs improper integrals of the *second kind*?) For which values of *p* does each of the following converge?



Two positive functions with $f(x) < g(x)$.

$(A)\ \int_1^\infty \frac{1}{x^p}\,dx$ $\qquad (B)\ \int_e^\infty \frac{1}{x(\ln x)^p}\,dx$ $\qquad (C)\ \int_3^\infty \frac{1}{x(\ln x)(\ln\ln x)^p}\,dx$

$(D)\ \int_0^\infty e^{-px}\,dx$

5. For each of the following improper integrals of the first kind, determine convergence or divergence. In each case, carefully explain how you obtained your answer.

$(A)\ \int_0^\infty \sin^2 x\,dx$ $\qquad (B)\ \int_2^\infty \frac{1}{x+\sin x}\,dx$

$(C)\ \int_{-\infty}^\infty \exp(-x^2)\,dx$ *Note:* exp(f(x)) means $e^{f(x)}$. $\qquad (D)\ \int_0^\infty \frac{9+91x^5+2018\sqrt{x}}{1+x^8}\,dx$

$(E)\ \int_0^\infty \frac{1+e^x}{1+x^{1000}}\,dx$ $\qquad (F)\ \int_2^\infty \frac{\cos^4 x}{x^2+x+1}\,dx$ $\qquad (G)\ \int_0^\infty \frac{1+e^{2x}}{1+e^{3x}}\,dx$

$(H)\ \int_0^\infty \frac{1+x+2x^2}{3+5x+9x^2+19x^3}\,dx$ $\qquad (I)\ \int_1^\infty \frac{\ln x}{x^3}\,dx$

$(J)\ \int_0^{\infty} \frac{x^2}{e^x}\,dx \qquad (K)\ \int_1^{\infty} \frac{1+e^{-x}}{x}\,dx \qquad (L)\ \int_e^{\infty} \frac{1}{\ln x}\,dx$

$(M)\ \int_1^{\infty} \frac{x^2+\ln x}{(\ln x)^4+x^2+\sqrt{x}+13}\,dx$

6. For which values of p does the following improper integral converge? $\int_{0+}^{1} \frac{1}{x^p}\,dx$

7. For each of the following improper integrals of the *second kind*, determine converge or divergence. In each case, carefully explain how you obtained your answer.

$(A)\ \int_{0+}^{1} \frac{11+x^2}{x^3}\,dx \qquad (B)\ \int_0^{1-} \frac{1}{\sqrt{1-x^2}}\,dx \qquad (C)\ \int_0^{\frac{\pi}{2}^-} \tan x\,dx$

$(D)\ \int_{0+}^{1} \ln\left(\frac{1}{x}\right)dx \qquad (E)\ \int_{0+}^{1} \frac{1+x+x^5}{x^9}\,dx$

8. How do *little oh* and *big oh* help us to implement the Comparison Test for improper integrals? Discuss the ***limit comparison*** test.

9. (Univ. Michigan, 11/2010). Each of the integrals below is improper. Determine the convergence or divergence of each. Make sure you include all the appropriate steps to justify your answers. Approximations with your calculator will not receive credit.

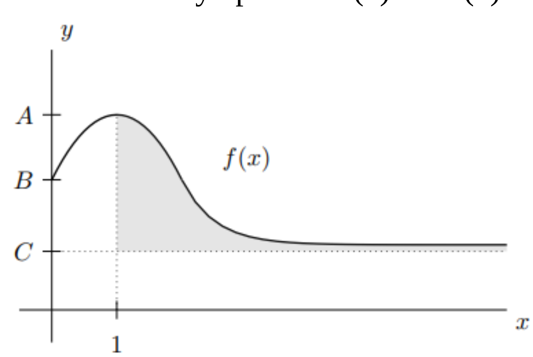
(a) $\int_1^{\infty} \frac{5-2\sin x}{\sqrt{x^3}}\,dx$ (b) $\int_2^{\infty} \frac{1}{(x^3+7)^{\frac{1}{3}}}\,dx$ (c) $\int_1^{2} \frac{x^2}{(x^3-1)^2}\,dx$

10. (Univ. Michigan, 11/2016). Determine whether the following integrals converge or diverge. If an integral converges, find its exact value (i.e., no decimal approximations) and write it in the blank provided. If it diverges, circle "DIVERGES" and explain why. In any case, show all your work, indicating any theorems you use, and using proper syntax and notation.

(a) $\int_0^{\infty} 2xe^{-cx}\,dx, where\ c > 0\ is\ a\ constant$

$(b)\ \int_0^{1} \frac{x}{\sqrt{x^5+x^7}}\,dx$

11. (Univ Michigan 4/2016) A function f has domain $[0, \infty)$, and its graph is given below. The numbers A, B, C are positive constants. The shaded region has finite area, but it extends infinitely in the positive x-direction. The line y = C is a horizontal asymptote of f(x) and f(x) > C for all x ≥ 0.



The point (1, A) is a local maximum of *f*.

(a) Determine the convergence of the improper integral. You must give full evidence supporting your answer, showing all your work and indicating any theorems about integrals you use. $\int_0^{1} \frac{f(x)}{x}\,dx$

(b) Circle the correct answer. The value of $\int_0^{\infty} f(x)f'(x)\,dx$

is $C-A$ is $\frac{C^2-A^2}{2}$ is $B-A$ cannot be determined diverges

(c) Circle the correct answer. The value of the integral $\int_1^{\infty} f'(x)dx$

is $C - A$ is $\frac{C^2 - A^2}{2}$ is C cannot be determined diverges

12. How does $\int_{-1}^{1} \frac{1}{x^2} dx$ serve as a "cautionary tale."