

Instructions: Answer any 16 of the following 18 questions. You may answer more than 16 to obtain extra credit. *You must show your reasoning*; calculator answers are not acceptable. Each problem is worth 6 points, unless otherwise stated.

1. Explain why the following improper integral converges.

$$\int_1^{\infty} \frac{x^2 - 1}{2x^5 + 3x + 17} dx$$

Solution: Since the integrand is roughly, $\frac{x^2}{2x^5} = \frac{1}{2} \frac{1}{x^3}$, we conjecture that our given integral converges (by virtue of the p-test for Type 1 improper integrals).

Using the Comparison Test, for $x \geq 1$, $0 \leq \frac{x^2 - 1}{2x^5 + 3x + 17} < \frac{x^2}{x^5} = \frac{1}{x^3}$ we see that our original integral converges.

2. Determine if the following improper integral converges or diverges. *Justify your answer!*

$$\int_{5+\sqrt{x-5}}^7 \frac{1}{\sqrt{x-5}} dx$$

Solution:

Since $\int_{5+\sqrt{x-5}}^7 \frac{1}{\sqrt{x-5}} dx$ appears to behave as $\int_{0+}^5 \frac{1}{\sqrt{x}} dx$, we conjecture via the p-test for type 2 improper integrals that our original integral converges.

Using the definition of the improper integral:

$$\begin{aligned} \int_{5+\sqrt{x-5}}^7 \frac{1}{\sqrt{x-5}} dx &= \lim_{a \rightarrow 5+} \int_a^7 \frac{1}{\sqrt{x-5}} dx = \lim_{a \rightarrow 5+} 2(x-5)^{\frac{1}{2}} \Big|_a^7 = \\ &= \lim_{a \rightarrow 5+} 2(x-5)^{\frac{1}{2}} \Big|_a^7 = 2 \lim_{a \rightarrow 5+} \left(2^{\frac{1}{2}} - (a-5)^{\frac{1}{2}} \right) = 2^{3/2} \end{aligned}$$

3. Consider the recursive sequence defined by

$$a_1 = 1 \text{ and } a_{n+1} = \sqrt{6 + a_n} \text{ for all } n \geq 1.$$

- (a) Find a_1, a_2, a_3 , and a_4 . Round your answers to the nearest hundredth.

Solution: We are given that $a_1 = 1$. So $a_2 = \sqrt{6 + a_1} = \sqrt{7} \approx 2.65$

Next $a_3 = \sqrt{6 + a_2} \approx \sqrt{8.65} \approx 2.94$; Finally, $a_4 = \sqrt{6 + a_3} \approx 2.99$

- (b) Find the $\lim_{n \rightarrow \infty} a_n$ assuming that this limit exists.

Solution: Let $L = \lim_{n \rightarrow \infty} a_n$. Then, since $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n$ and $\lim_{n \rightarrow \infty} a_{n+1} = \sqrt{6 + \lim_{n \rightarrow \infty} a_n}$, we have:

$L = \sqrt{6 + L}$ which implies that $L^2 - L - 6 = 0$. Factoring: $(L - 3)(L + 2) = 0$, so $L = -2$ or $L = 3$.

As L is defined to be a limit of positive numbers, we reject the possibility that $L = -2$.

Thus $L = 3$.

For each of the following *sequences*, determine convergence or divergence. In the case of divergence, find the limit of the sequence. *Show your work.*

$$4. a_n = n \sin \frac{1}{n}$$

Solution: Let $h = 1/n$. Then $n = 1/h$ and as $n \rightarrow \infty$, $h \rightarrow 0$. Hence:

$$a_n = n \sin \frac{1}{n} = \frac{\sin h}{h} \rightarrow 1 \text{ as } h \rightarrow 0$$

$$5. b_n = \int_0^n e^{-5x} dx$$

$$\text{Solution: } b_n = \int_0^n e^{-5x} dx = \left(-\frac{1}{5} e^{-5x} \Big|_0^n \right) = -\frac{1}{5} (e^{-5n} - 1) \rightarrow \frac{1}{5}$$

$$6. c_n = \frac{1}{\arctan(\ln(\ln n))}$$

$$\text{Solution: Since } \ln \ln n \rightarrow \infty \text{ as } n \rightarrow \infty, \quad c_n = \frac{1}{\arctan(\ln(\ln n))} \rightarrow \frac{1}{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$7. d_n = \frac{n(5n+1)(n-2018)^5}{1789 + \ln n + (2n+3)^3(n-11)^4}$$

Solution:

$$d_n = \frac{n(5n+1)(n-2018)^5}{1789 + \ln n + (2n+3)^3(n-11)^4} \approx \frac{n(5n)n^5}{(2n)^3 n^4} = \frac{5}{8}$$

$$8. e_n = \sqrt{n^2 + 5n} - \sqrt{n^2 - 21n}$$

Solution: Rationalizing the “numerator” yields:

$$e_n = \sqrt{n^2 + 5n} - \sqrt{n^2 - 21n} = \left(\sqrt{n^2 + 5n} - \sqrt{n^2 - 21n} \right) \frac{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}} =$$

$$\frac{(n^2 + 5n) - (n^2 - 21n)}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}} = \frac{26n}{\sqrt{n^2 + 5n} + \sqrt{n^2 - 21n}} \rightarrow 13$$

$$9. \quad z_n = \int_2^n \frac{1}{\sqrt{x}-1} dx$$

Solution: Let $x = u^2$; then $dx = 2u du$. So

$$z_n = \int_2^n \frac{1}{\sqrt{x}-1} dx = \int_{\sqrt{2}}^{\sqrt{n}} \frac{1}{u-1} 2u du = 2 \int_{\sqrt{2}}^{\sqrt{n}} \frac{u}{u-1} u du$$

Next, let $y = u - 1$; $dy = du$. So

$$z_n = 2 \int_{\sqrt{2}}^{\sqrt{n}} \frac{u}{u-1} du = 2 \int_{\sqrt{2}}^{\sqrt{n}} \frac{y+1}{y} dy = 2 \int_{\sqrt{2}}^{\sqrt{n}} \left(1 + \frac{1}{y}\right) dy =$$

$$2(y + \ln y) \Big|_{\sqrt{2}}^{\sqrt{n}} = 2(\sqrt{n} + \ln \sqrt{n} - \sqrt{2} - \ln \sqrt{2}) = \infty.$$

Hence, the sequence diverges.

10. [2 pts each] For each of the following statements answer **True or False**. Briefly justify each answer!

(a) $x^3 \ln x + x + 1 = o(x^4)$

Answer: True

(b) $4^n = o(\pi^n)$

Answer: False

(c) $\frac{3x^3(x^2+1)^5 + 5x \ln x + 99}{x^5 + 5x^3 + x + 2015} = O(x^8)$

True since:

$$\frac{3x^3(x^2+1)^5 + 5x \ln x + 99}{x^5 + 5x^3 + x + 2015} = \frac{3x^3(x^2+1)^5 + 5x \ln x + 99}{x^{13} + 5x^{11} + x^9 + 2015x^8} \rightarrow 3 \text{ as } x \rightarrow \infty$$

(d) $(\ln x)^{2018} = o(x)$

Answer: True

$$(e) \left(1 + \frac{1}{x}\right)^{3x} = o(e^{4x})$$

Answer: True, since $\left(1 + \frac{1}{x}\right)^{3x} \rightarrow e^3$

11. Evaluate $\int \arcsin x \, dx$

Solution: Using integration by parts:

$$\text{Let } f(x) = \arcsin x$$

$$\text{Then } g'(x) = 1$$

$$\text{So } g(x) = x \text{ and } f'(x) = \frac{1}{\sqrt{1-x^2}}$$

Thus

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx = x \arcsin x - \sqrt{1-x^2} + C$$

12. Evaluate

$$\int x^{\frac{1}{4}} \ln x \, dx$$

Solution: Use substitution:

$$\text{Let } u = x^{\frac{1}{4}}; \text{ then } x = u^4 \text{ and so } dx = 4u^3 \, du$$

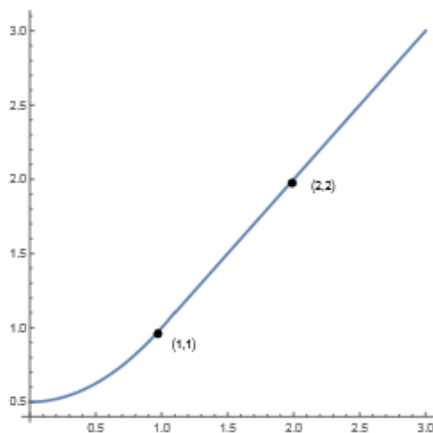
Thus

$$\int x^{\frac{1}{4}} \ln x \, dx = \int u \ln(u^4) 4u^3 \, du = \int u 4 (\ln u) 4u^3 \, du = 16 \int u^4 \ln u \, du =$$

$$\frac{4}{25} x^{\frac{5}{4}} (5 \ln x - 4) + C \text{ since}$$

$$\int u^4 \ln u \, du = \frac{1}{25} u^5 (5 \ln u - 1)$$

13. The graph of the function $p(x)$ is shown below:



Use integration by parts with the selection of $f = x$ and $g' = p''(x) dx$ along with information from the graph to find the value of $\int_0^2 x p''(x) dx$.

Solution: Integration by parts formula

$$\int_0^2 f(x)g'(x)dx = f(x)g(x) \Big|_0^2 - \int_0^2 f'(x)g(x)dx$$

Now let $f(x) = x$ and $g'(x) = p''(x)$.

Then $f'(x) = 1$ and $g(x) = p'(x)$.

$$\text{So } \int_0^2 x p''(x) dx = f(x)g(x) \Big|_0^2 - \int_0^2 f'(x)g(x) dx = f(2)g(2) - f(0)g(0) - \int_0^2 1 p'(x) dx =$$

$$2p'(2) - 0 - \left(p(x) \Big|_0^2 \right) = 2p'(2) - (p(2) - p(0)) = 2p'(2) - p(2) + p(0) = 2(1) - 2 + 0.5 = \frac{1}{2}$$

Here we have estimated $p'(2)$ using the fact that the graph is virtually linear from (1, 1) to (2, 2).

For each improper integral given below, determine *convergence* or *divergence*. (You will need to use the Comparison Test here.) *Justify your answers!*

14. $\int_4^{\infty} \frac{1}{(\ln x - 1)^2} dx$

Solution:

Note that $\frac{1}{(\ln x - 1)^2} > \frac{1}{(\ln x)^2} > \frac{1}{x} > 0$.

Now $\int_4^{\infty} \frac{1}{x} dx$ diverges; so by the Comparison Test,

$\int_4^{\infty} \frac{1}{(\ln x - 1)^2} dx$ diverges as well.

$$15. \int_{0+}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx$$

Solution:

$$\int_{0+}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx = \int_{0+}^1 \frac{x+3}{\sqrt{x^3+x^5}} dx + \int_1^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx$$

$$\text{Now, for } 0 < x < 1, \frac{x+3}{\sqrt{x^3+x^5}} > \frac{3}{\sqrt{x^3}} = 3 \frac{1}{x^{\frac{3}{2}}} > 0$$

Now, invoking the p -test for type 2 improper integrals,

$$\int_{0+}^1 \frac{x+3}{\sqrt{x^3+x^5}} dx \text{ diverges}$$

$$\text{Hence } \int_{0+}^{\infty} \frac{x+3}{\sqrt{x^3+x^5}} dx \text{ diverges.}$$

16. Evaluate the integrals below, given that $f(x)$ is a continuous function for $0 \leq x \leq 6$ with the following properties:

$$f(0) = 2, \quad f(2) = 3, \quad f(4) = -1, \quad f(6) = 5; \quad f'(0) = 1, \quad f'(2) = 4;$$

$$\int_0^2 f(x) dx = 3, \quad \int_2^4 f(x) dx = 1, \quad \int_4^6 f(x) dx = 6.$$

$$\text{(a) } \int_0^2 x f'(x) dx = 3.$$

Solution: Integration by parts.

Let $u(x) = x$ and $v'(x) = f'(x)$. Then $u'(x) = 1$ and $v(x) = f(x)$.

$$\text{So } \int_0^2 u(x)v'(x) dx = u(x)v(x) \Big|_0^2 - \int_0^2 u'(x)v(x) dx =$$

$$2f(2) - \int_0^2 f(x) dx = 2(3) - 3 = 3$$

$$(b) \int_2^4 f'(x)(2 + 3f(x)) dx = -20.$$

Break up the integral into two pieces. The first one is done using the FTC directly, while the second one is handled by noting $f'(x)f(x) = \frac{1}{2}(f^2(x))'$.

$$\begin{aligned} \int_2^4 f'(x)(2 + 3f(x)) dx &= 2 \int_2^4 f'(x) dx + \frac{3}{2} \int_2^4 (f^2(x))' dx \\ &= 2(f(4) - f(2)) + \frac{3}{2}(f(4)^2 - f(2)^2) = -20. \end{aligned}$$

$$(c) \int_0^2 f(3x) dx = \frac{10}{3}.$$

By substitution, setting $u = 3x$, so $dx = du/3$ and the new limits of integration 0 and 6, we find:

$$\int_0^2 f(3x) dx = \frac{1}{3} \int_0^6 f(u) du = \frac{1}{3} \left(\int_0^2 f(u) du + \int_2^4 f(u) du + \int_4^6 f(u) du \right) = \frac{3 + 1 + 6}{3} = \frac{10}{3}.$$

17. Consider a group of people who have received a new treatment for pneumonia. Let

t be the number of days it takes for a person with pneumonia to fully recover. The probability density function giving the distribution of t is

$$f(t) = \frac{10}{(1 + at)^2}, \quad \text{for } t > 0,$$

for some positive constant a .

- (a) Give a practical interpretation of the quantity $\int_3^{10} f(t) dt$. You need not compute the value of this integral. Use complete sentences.

Solution: The fraction of the people with pneumonia who recovered during a period of three to ten days after treatment.

- (b) Find a formula for the *cumulative distribution function* $F(t)$ of $f(t)$ for $t > 0$. Show all your work. Your answer may include the constant a . Your final answer should not include any integrals.

$$\text{Solution: } F(t) = \int_0^t \frac{10}{(1 + ax)^2} dx = -\frac{10}{a(1 + ax)} \Big|_0^t = \frac{10}{a} - \frac{10}{a(1 + at)}$$

- (c) Determine the value of a . Show your work.

Solution: Since $f(t)$ is a probability density function, then $1 = \int_0^{\infty} \frac{10}{(1+ax)^2} dx$. Hence

$$\int_0^{\infty} \frac{10}{(1+ax)^2} dx = \lim_{b \rightarrow \infty} \int_0^b \frac{10}{(1+ax)^2} dx = \lim_{b \rightarrow \infty} \frac{10}{a} - \frac{10}{a(1+ab)} = \frac{10}{a}.$$

Hence $a = 10$.

18. (*Univ Michigan*) The lifetime t (in years) of a tree has probability density function

$$f(t) = \begin{cases} \frac{a}{(t+1)^p} & \text{for } t \geq 0. \\ 0 & \text{for } t < 0. \end{cases}$$

where $a > 0$ and $p > 1$.



(a) Use the comparison method to find the values of p for which the average lifetime M is finite ($M < \infty$).

Properly justify your answer.

Solution: The average lifetime M is given by the formula $M = \int_0^{\infty} t \frac{a}{(t+1)^p} dt$.

Since

$$t \frac{a}{(t+1)^p} \leq t \frac{a}{t^p} = \frac{a}{t^{p-1}} \quad \text{for } t > 0,$$

then

$$\int_1^{\infty} t \frac{a}{(t+1)^p} dt \leq \int_1^{\infty} \frac{a}{t^{p-1}} dt$$

We know that $a \int_1^{\infty} \frac{1}{t^{p-1}}$ converges precisely when $p-1 > 1$ ($p > 2$) by the p -test, so the first integral converges precisely when $p > 2$. This implies that the average lifetime M is finite for $p > 2$.

(b) Find a formula for a in terms of p . Show all your work.

Solution: We know that

$$1 = \int_0^{\infty} \frac{a}{(t+1)^p} dt.$$

We use u -substitution with $u = t + 1$ to calculate the integral:

$$\begin{aligned} \int_0^{\infty} \frac{a}{(t+1)^p} dt &= \lim_{b \rightarrow \infty} \int_0^b \frac{a}{(t+1)^p} dt \\ &= \lim_{b \rightarrow \infty} \int_1^{b+1} \frac{a}{u^p} du = a \lim_{b \rightarrow \infty} \int_1^{b+1} u^{-p} du \\ &= a \lim_{b \rightarrow \infty} \frac{u^{-p+1}}{(-p+1)} \Big|_1^{b+1} = a \lim_{b \rightarrow \infty} \frac{1}{(-p+1)u^{p-1}} \Big|_1^{b+1} \\ (\text{since } p > 1) &= \frac{a}{p-1}. \end{aligned}$$

Therefore $1 = \frac{a}{p-1}$, so $a = p - 1$.

(c) Let $C(t)$ be the cumulative distribution function of $f(t)$. For a given tree, what is the practical interpretation of the expression $1 - C(30)$?

Solution: $1 - C(30)$ is the probability that a given tree lives at least 30 years.

Extra Credit A:

[MIT integration bee]

$$\int e^{\arccos x} dx$$

Hint: Begin with a u substitution.

Solution: Let $u = \arccos x$. Then $x = \cos u$, and $dx = -\sin u$.

So

$$\int e^{\arccos x} dx = - \int e^u \sin u du$$

Using integration by parts (twice) we find that $\int e^u \sin u du = \frac{1}{2} e^u (\sin u - \cos u) + C$.

Note that $\cos u = x$ and $\sin u = \sqrt{1 - \cos^2 u} = \sqrt{1 - x^2}$.

Finally,

$$\int e^u \sin u du = \frac{1}{2} e^u (\sin u - \cos u) + C = \frac{1}{2} e^{\arccos x} (x - \sqrt{1 - x^2}) + C$$

If a lion could talk, we could not understand him.

- **Ludwig Wittgenstein**