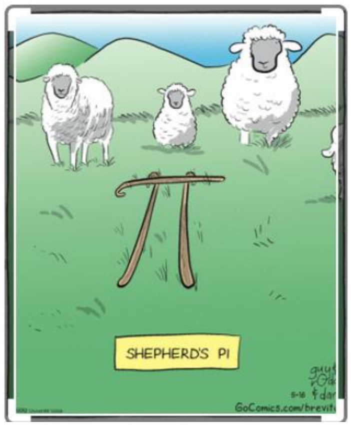
MATH 162 PRACTICE TEST II

March 2020



1) For each given sequence, determine *convergence* or *divergence*. If the sequence converges, find its limit. Justify your answers.

(a) $a_n = \dfrac{100^n + 1789^n}{n! + 7^n}$ (b) $b_n = \left(1 + \frac{1}{3n}\right)^n$ (c) $c_n = \dfrac{\ln(n + 2020\pi)}{\ln n}$

(d) $d_n = \dfrac{\cos\left(\frac{\pi}{n}\right)}{n}$ (e) $e_n = \displaystyle\int_0^n e^{-\pi t}\,dt$ (f) $f_n = \sqrt{\dfrac{n+1}{n} + \dfrac{\sin(n^2)}{n^2} + e^{-\frac{3}{n}} + \dfrac{e^n}{\sinh n}}$

(g) $g_n = \dfrac{(\ln \ln n)^{2525}}{n}$

2) Give an example of two *divergent* numerical sequences whose sum is *convergent.*

3) Consider the following recursively defined sequence: $c_1 = 7$, $c_2 = 4$, and

$$c_{n+1} = \frac{\left(c_n + \dfrac{5}{(c_{n-1})^2}\right)}{2} \quad \textit{for } n \geq 2$$

(a) Find the values of c_3, c_4 and c_5.
(b) Assuming that the limit of c_n exists, find its value.

4) *Find* $\displaystyle\lim_{n\to\infty} n^{\frac{1}{\ln n}}$ *(Show your work!)*

5) For each of the following sequences, determine convergence or divergence. In the case of convergence, find the limit of the sequence.

(a) $x_n = e^{\frac{1}{n}}$ (b) $y_n = \dfrac{n!}{n+1}$ (c) $z_n = \dfrac{\sin n}{n} + \dfrac{5}{n}$

(d) $c_n = \dfrac{3(2n+1)^3}{(1-n)^2(4n+13)}$ (e) $a_n = \sec\left(\ln\left(\sin^4\left(\dfrac{\pi}{2} + \dfrac{1}{n^2}\right)\right)\right)$

6) For each of the following infinite series, determine *convergence* or *divergence.* In the case of convergence, *find the sum of the series:*

(a) $\sum_{n=1}^{\infty} \ln \frac{n+1}{n}$ (b) $\sum_{n=0}^{\infty} \frac{5}{9^n}$ (c) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$

(d) $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (e) $\sum_{n=1}^{\infty} \cos\left(\frac{5}{n}\right)$ (f) 0.129912991299…

7) For each series below, determine absolute convergence, conditional convergence, or divergence. Justify each answer.

(a) $\sum_{n=3}^{\infty} (-1)^n \frac{13}{(\ln n)^{13}}$ (b) $\sum_{k=1}^{\infty} (-1)^k \frac{(k+3)(k^2+5)}{(k+13\ln k)^4}$

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{\left(1+\frac{1}{n}\right)^{n^2}}$ (d) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln(e^n+e^{-n})}$ (e) $\sum_{n=1}^{\infty} (-1)^n \frac{n^{13}}{(n+13)!}$

8) For each power series below, determine the *radius of convergence* and the *interval of convergence*. Study the behavior of each power series at the *endpoints*.

(a) $\sum_{n=1}^{\infty} \frac{13^n}{n(n+13)} x^n$ (b) $\sum_{n=1}^{\infty} \frac{1}{n(n+3)(n+11)} (x-4)^n$ (c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n+7}} (x+13)^n$

9) (a) Find the 3rd order Maclaurin polynomial of $\cosh x = \frac{1}{2}(e^x + e^{-x})$

(b) Find the 5th order Taylor polynomial of $\cos x$ centered at $x = \pi/2$.

10) Find the 4th order Taylor polynomial of e^x centered at x = 2.

11) Find the 3rd order Maclaurin polynomial of

$f(x) = 4 + (x+13)^2 + (x+13)^3$

12) By differentiating the power series expansion of $y = \frac{1}{1-x}$, find the value of

$$\sum_{k=0}^{\infty} \frac{k}{13^k}$$

13) Find the *first five* non-zero terms of the Maclaurin series expansion of

$$h(x) = (1 + 2x^2)\, e^{3x}.$$

14) Let $f(x) = x^8 e^{5x}$. Compute $f^{(100)}(0)$. Do not simplify your answer.

15) Find the *radius of convergence* of the power series:

$$\sum_{n=1}^{\infty} \frac{n!}{(1)(3)(5)(7)...(2n-1)} x^n$$

16) Find the *radius of convergence* of the power series:

$$\sum_{n=0}^{\infty} n!\, x^{2^n}$$

17) Find the *radius of convergence* of the power series: $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n} x^n$

18) *Without using* l'Hôpital's rule, calculate the following limit. Show your work!

$$\lim_{t\to 0} \frac{te^{4t} - \sin(3t) + 2t - 4t^2}{t^3}$$

19) Let $G(x) = x^3 \cosh(3x)$. Using an appropriate Maclaurin series, compute $G^{(2020)}(0)$. (Do not try to simplify your answer.)

20) Find the first four non-zero terms of the Maclaurin series for each of the following:

$$(a)\ \frac{e^{2x}}{\cosh x} \qquad (b)\ \frac{\ln(1+x)}{1+x^2} \qquad (c)\ e^{x^2}\sin 2x$$

21) Using *division* of power series, find the first three non-zero terms of the Maclaurin series expansion of

$$f(x) = \frac{e^{2x}+1}{\cos x}$$

22) Using multiplication of power series, find the first four non-zero terms of the Maclaurin series expansion of

$$g(x) = e^{x^2}(1 + x^2 + x^3)$$

23) Determine the *interval of convergence* of the following power series. (You need not study end-point behavior.)

$$\sum_{n=1}^{\infty} \frac{n^{13}\, 13^n}{\sqrt{n+2016}} (x-13)^n$$

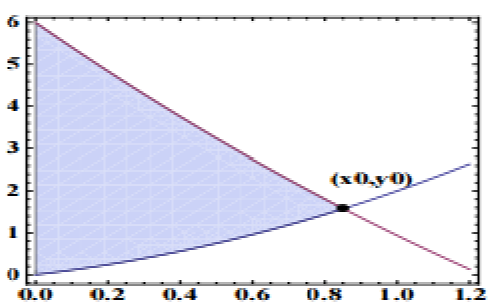
24) Analyze the behavior of the series

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^2+4}}{\left(n^{1/3}+1789\right)^5}$$

25) What is the relationship between cosh x and cos x? between sinh x and sin x?

State *Euler's formula*.

26) *[University of Michigan final exam question]* The graph shows the area between the graphs of $f(x) = 6\cos(\sqrt{2x})$ and $g(x) = x^2 + x$. Let (x_0, y_0) be the intersection point between the graphs of f(x) and g(x).



a) Compute P(x), the function containing the first three nonzero terms of the Taylor series about x = 0 of $f(x) = 6\cos(\sqrt{2x})$.

b) Use P(x) to approximate the value of x_0.

c) Use P(x) and the value of x_0 you computed in the previous question to write an integral that approximates the value of the shaded area.
d) Graph f(x) and g(x) in Wolfram Alpha or your calculator. Use the graphs to find an approximate value for x_0.
e) Write a definite integral in terms of f(x) and g(x) that represents the value of the shaded area. Find its value using Wolfram Alpha or your calculator.

27) *[University of Michigan final exam question]* (a) Find the Maclaurin series of $\sin(x^2)$. Your answer should include a formula for the general term in the series.
(b) Let *m* be a positive integer, find the Maclaurin series of $\cos(m\pi x)$. Your answer should include a formula for the general term in the series.
(c) Use the second degree Maclaurin polynomials of $\sin(x^2)$ and $\cos(m\pi x)$ to approximate the value of b_m, where

$$b_m = \int_{-1}^{1} \sin(x^2)\cos(m\pi x)\,dx$$

(The number b_m is called a *Fourier coefficient of the function sin(x^2).* These numbers play a key role in Fourier analysis, a subject with widespread applications in engineering and the sciences.)

Jean-Baptiste Joseph Fourier (1768 – 1830)