**1.** *[10 pts]* List the following 13 functions in increasing order of their growth rates as $n \to \infty$. In case of a tie, list the functions on the same line. For example, $3n^2$ and $5n^2$ have the same order of magnitude.

$$\sqrt[3]{n^6}, \quad n, \quad \frac{e^{2n}}{1234567}, \quad \ln\ln n, \quad e^n, \quad n\ln n, \quad \left(\frac{n^7+2020}{n^6+1}\right)^2$$

$$ln\sqrt{n^2+n+1}, \quad n^5+\ln n, \quad e^{\ln n}, \quad \sqrt{n}, \quad 2020\,n$$

*Solution:* Note that:

(1) $\sqrt[3]{n^6}$ *is asymptotic to* $n^2$.

(2) $\frac{e^{2n}}{1234567}$ is *asymptotic to* $e^{2n}$

(3) $\left(\frac{n^7+2020}{n^6+1}\right)^2$ *is asymptotic to* $n^2$.

(4) $ln\sqrt{n^2+n+1}$ *is asymptotic to* $\ln n$.

(5) $n^5+\ln n$ *is asymptotic to* $n^5$

(6) $n$ *is equal to (and thus asymptotic) to* $e^{\ln n} = n$

(7) $2020\,n$ *is asymptotic to* $n$

Hence, the line-up, in increasing order is:

$\ln\ln n$

$ln\sqrt{n^2+n+1}$

$\sqrt{n}$

n, *same growth rate as* $e^{\ln n}$

$n\ln n$

$\sqrt[3]{n^6}$ *same growth rate as* $\left(\frac{n^7+2020}{n^6+1}\right)^2$ *or* $n$.

$n^5+\ln n$

$e^n$

$\frac{e^{2n}}{1234567}$

**2.** *[10 pts]* Suppose $b$ is a positive constant. Find the exact value of the convergent improper integral $\int_0^\infty e^{-bx}dx$

*Solution:* *Evaluating the Riemann integral* $\int_0^c e^{-bx}dx = -\frac{1}{b}\left(e^{-bc}-e^0\right) = \frac{1}{b}\left(1-e^{-bc}\right)$

Next, letting $c \to \infty$ *and noting that* $b > 0$, $\frac{1}{b}(1 - e^{-bc}) \to 0$

Hence $\int_0^{\infty} e^{-bx} dx = \lim_{c\to\infty}\left(\int_0^c e^{-bx} dx\right) = 1.$

**3.** *[10 pts]* Consider the improper integral given below. Determine whether it is convergent or divergent. If it is convergent, evaluate.

Note: Do not use a more general result from class to answer this question.

$$\int_e^{\infty} \frac{1}{x\sqrt{\ln x}} dx$$

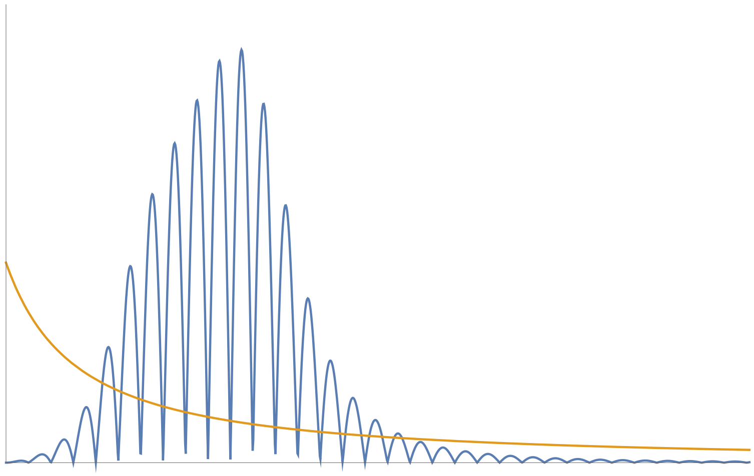
*Solution:* Note that the usual p-test does not apply here. Instead, we evaluate the integral.

$$\int_e^{c} \frac{1}{x\sqrt{\ln x}} dx = \int_e^{c} (\ln x)^{-\frac{1}{2}} \frac{1}{x} dx = 2(\ln x)^{\frac{1}{2}} \Big[_e^c = 2\left(\sqrt{\ln c} - \sqrt{\ln e}\right) = 2\left(\sqrt{\ln c} - 1\right) \to \infty$$

Thus, the improper integral diverges.

Extra Credit

Below are graphs of the function $f(x) = \frac{1}{(x+0.9)^{0.9}}$ and a mystery function $g(x)$ satisfying $g(0) = 0$.



(a) *[1 pt]* On the graph, label which is $f(x)$ and which is $g(x)$. *Explain* your answer in the space below.

**Answer:** *The function* $y = \frac{1}{(x+0.9)^{0.9}}$ *is a hyperbola that is decreasing.*

*Hence the oscillating function must be* $g$.

(b) *[3 pts]* Based on the graph, determine if it is possible to tell whether the improper integral $\int_0^{\infty} g(x)dx$ converges or diverges. Justify your answer.

**Answer:** No, it is not possible to tell if the oscillating function converges or diverges.
Note that the given function, $f(x) = \frac{1}{(x+0.9)^{0.9}}$ is just a horizontal translation of the more familiar curve $h(x) = \frac{1}{x^{0.9}}$. By the p-test, $\int_0^\infty h(x)dx\ diverges, and\ so\ must\ \int_0^\infty f(x)dx$.
Now, we cannot use the comparison test, since g(x) lies below the graph of f(x) for large x.