MATH 201 SOLUTIONS: TEST I 18 FEBRUARY 2020

"Don't count the days; make the days count."

— **Muhammad Ali**

1. Consider the universe of all people in the world. We define the following predicates:

E(x) = "x is a European" S(x) = "x is smart" A(x, y) = "x is angry at y"

Express each of the following in symbolic form.

(a) All Europeans are smart.

$$\forall x, E(x) \Rightarrow S(x)$$

Equivalently, $\forall x, \sim E(x) \vee S(x)$

(b) There exists a smart European.

$$\exists x, E(x) \wedge S(x)$$

(c) Every European is angry at some other European

$$\forall x, E(x) \Rightarrow \exists y \left(\sim(x = y)\right) \wedge E(y) \wedge A(x, y)$$

(d) No one is angry at anyone.

$$\forall x\, \forall y \sim A(x, y)$$

2. A coin is tossed 13 times in a row.

(a) How many possible sequences of heads and tails are there?

Answer: 2^{10}

(b) In how many ways can there be exactly 5 heads?

Answer: $\binom{13}{5}$

(c) In how many ways can there be precisely 5 heads such that no two heads be adjacent?

Answer: 5 Heads and 8 Tails. We use the method of stars and bars. There are 8 stars and 4 bars.

Thus $\binom{9}{4}$

3. (a) Fifteen *indistinguishable* marbles are to be given to 6 children. In how many ways can this be achieved?

Answer: Here we have 15 stars and 5 bars. The four bars will represent barriers between adjacent children. Hence $\binom{20}{5}$

(b) Fifteen *distinguishable* marbles are to be given to 6 children. In how many ways can this be achieved?

Answer: Each marble must choose one child from six. Hence 615

4. Suppose that |A| = k and |B| = n. Find the cardinality of each of the following sets:

(a) $\mathcal{P}(\mathcal{P}(A))$

Answer: 2^{2^k}

(b) $\mathcal{P}(A \times B)$

Answer: 2^{kn}

(c) $\{X \in \mathcal{P}(A): |X| = 1\}$

Answer: k

(d) $\{X \subseteq \mathcal{P}(A): |X| = 1\}$

Answer: 2^k

5. If you are dealt a hand of 8 cards from a standard deck (without regard to order), in how many ways can you have:

(a) A flush? (all 8 from the same suit)

Answer: $\binom{4}{1}\binom{13}{8}$

(b) 4 distinct pairs? (no more than 2 cards of the same rank)

Answer: $\binom{13}{4}\binom{4}{2}^4$

(c) No two of a kind. (excludes three or four of a kind)

Answer: *Choose eight ranks; then choose one card from each of the eight ranks.*

$$\binom{13}{8}4^8$$

6. Let A, B, and C be sets. Each statement below is false. For each false statement, provide a *specific* counterexample.

(a) $(\overline{A \cup B}) = \overline{A} \cup \overline{B}$

Answer: *Let X, the universe, be {1, 2, 3, 4}. Let A = {1} and let B = {2}.*

Then $(\overline{A \cup B}) = \overline{\{1,2\}} = \{3,4\}$ *and* $\overline{A} \cup \overline{B} = \{2,3,4\} \cup \{1,3,4\} = \{1,2,3,4\}$.

(b) $(A \cap B) - C = (A - C) \cup (B - C)$

Answer: *Let X, the universe, be N. Let* $A = \{1\}, B = \{2\}, and\ C = \emptyset$.

Then $A \cap B = \emptyset$ *and so* $(A \cap B) - C = \emptyset$ *and*

$(A - C) \cup (B - C) = A \cup B = \{1, 2\}$

7. Let P be the statement, "I loved the film *Parasite*."

Let Q be the statement, "I don't swim in the lake."

Let R be the statement, "I am a dolphin."

Let S be the statement, "I am studying Arabic."

Express each of the following as a clearly written English sentence.

(a) $P \Rightarrow Q$

Answer: If I loved the film Parasite, then I don't swim in the lake.

(b) $(P \vee S) \wedge \sim(P \wedge S)$

Answer: I loved the film Parasite or I am studying Arabic, but not both.

(This is an *exclusive* or.)

(c) $(S \wedge R) \Rightarrow \sim P$

Answer: If I am a dolphin studying Arabic, then I did not love the film Parasite."

8. Write each of the following as an English sentence. State whether the original sentence is true or false.

(a) $\forall n \in \boldsymbol{N}, \exists X \in \mathcal{P}(\boldsymbol{N}),\ |X| < n$

Answer: For every natural number, there exists a subset of N of cardinality smaller than n.

This is a true statement.

(b) $\forall n \in \boldsymbol{Z}, \exists m \in \boldsymbol{Z}, m = n + 5$

Answer: For every integer n, there exists an integer m for which m = n + 5.

This statement is true. Since $n + 5 \in \boldsymbol{Z}$

(c) $\exists m \in \boldsymbol{Z}, \forall n \in \boldsymbol{Z}, m = n + 5$

Answer: There exists an integer m that has the property that for any integer n, m = n + 5.

This statement is false.

9. *Negate* each of the following statements: Your answer should be a sentence in English – not a symbolic sentence. (Nor should you write: "It is not true that ….").

(a) The number x is positive, but the number y is not positive.

Answer: Using deMorgan's law: The number x is not positive, or the number y is positive.

(b) For every prime number p, there is another prime number q with q > p.

Answer: *There is a prime number p such that for any prime number q,* $q \leq p$

(c) You cannot teach a cat to fetch, but you can teach a dog to swim.

Answer: *You can teach a cat to fetch or you cannot teach a dog to swim.*

(d) **(extra credit)** If the cat ate the mouse or the mouse ate the flea, then Albertine will either watch Dark on Netflix or go swimming in the lake.

Answer: If the cat ate the mouse or the mouse ate the flea, then Albertine will not watch Dark on Netflix and will not go swimming.

10. Let *U* be the universe of all positive integers greater than 2.

Consider the following predicates:

P(x) = "x is a prime number" Q(x) = "x is odd"

Express each of the following statements in symbolic form.

(a) "x being prime is a *sufficient* condition for x being odd."

Answer: $\forall x \ P(x) \Rightarrow Q(x)$

(b) "x being odd is a *necessary* condition for x being prime."

Answer: $\forall x, \ P(x) \Rightarrow Q(x)$

(c) "if x is odd and y is prime, then x + y is not odd."

Answer: $\forall x \, \forall y, \ (Q(x) \wedge P(y)) \Rightarrow {\sim}Q(x+y)$