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1 INTRODUCTION

My area of research is algebraic topology, in particular, surgery theory of manifolds. More specifically my work deals with the non multiplicativity of the signature of fibre bundles, and its relation to the Arf and Brown-Kervaire invariants.

Under certain conditions the signature of the total space in a fibration $F^m \to E^{4k} \to B^n$ of closed geometric Poincaré complexes satisfying Poincaré duality can be expressed as the product of the signatures of the base space and the fiber. Nevertheless this is not always the case, so that in general we have

$$\sigma(E) \neq \sigma(B)\sigma(F).$$

The problem of the non-multiplicativity of the signature has been investigated using many different methods of research. The use of index theory, spectral sequences [CHS57, Mor71], group cohomology [Mey73], K-theory [LR88, HKR07] and L-theory [LR88, LR92, Kor05] has shed light on different aspects of the problem, but there are still many unanswered questions in this field.

Chern, Hirzebruch and Serre [CHS57] were the first to consider the problem of non-multiplicativity of the signature of a fibre bundle. They determined that if the fundamental group of the base $\pi_1(B)$ acts trivially on the cohomology ring of the fibre $H^*(F, \mathbb{Q})$, then the signature is multiplicative. An interesting idea to investigate is:

To what extent is the signature multiplicative when $\pi_1(B)$ acts non trivially on the cohomology of the fibre.

Kodaira, Hirzebruch and Atiyah independently constructed examples of fiber bundles with non-multiplicative signature and non-trivial action of the fundamental group.

Meyer [Mey73] proved that the signature of a surface bundle is in general divisible by 4. Later on, Hambleton, Korzeniewski and Ranicki [HKR07] provided a high-dimensional version of this result. They proved that with $F \to E \to B$ a fibre bundle of closed, connected, compatibly oriented manifolds,

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod 4.$$

In my work, I deal with the following question:

To what extent is the signature multiplicative when the action of $\pi_1(B)$ is trivial only on the middle dimensional cohomology of the fibre with $\mathbb{Z}_2$ coefficients.

This question gives rise to interesting results concerning the signature modulo 8,

$$\sigma(E) \equiv \sigma(B)\sigma(F) \pmod 8.$$

My research interests fall into three main directions:

(1) Investigate further the relationship of the signature of a fibre bundle with other invariants like the Arf invariant and the Brown-Kervaire invariant extending the work initiated in my PhD thesis.

(2) Produce examples of bundles with non-multiplicative signature in high dimensions.

(3) Investigate the relation of the non-multiplicativity of the signature with TQFTs and give a precise interpretation of the theory in [Ban13].

2 Summary of Previous and Current Work

In my thesis I concentrate in questions relating to the signature modulo 8 of a fibration.

2.1 A new proof of Morita's theorem

A very relevant result in the context of the signature modulo 8 is a Theorem of Morita [Mor71, Thm 1.1]. This Theorem states that the signature modulo 8 of a $4k$-dimensional Poincaré complex X is the Brown-Kervaire invariant of the $\mathbb{Z}_4$-valued Pontryagin square on the $\mathbb{Z}_2$-vector space $H^{2k}(X;\mathbb{Z}_2)$.

The $\mathbb{Z}_8$-valued Brown-Kervaire invariant classifies $\mathbb{Z}_4$-valued quadratic forms up to isomorphism. With V a $\mathbb{Z}_2$ vector space and $\lambda : V \otimes V \to \mathbb{Z}_2$ a non-singular symmetric pairing, let $h : V \to \mathbb{Z}_4$ be a quadratic enhancement of the symmetric form so that

$$h(x+y) = h(x) + h(y) + j\lambda(x \otimes y) \text{ for all } x, y \in V$$

with $j : \mathbb{Z}_2 \to \mathbb{Z}_4$ the non-trivial homomorphism.

Definition 2.1. *The Brown-Kervaire invariant* $\mathrm{BK}(V,h)$ *of a quadratic form* $h : V \to \mathbb{Z}_4$ *is defined using a Gauss sum,*

$$\sum_{x\in V} i^{h(x)} = \sqrt{|V|}e^{2\pi i \mathrm{BK}(V,h)/8}$$

with $i^2 = -1$ *and* $x \in V$.

The original statement of [Mor71, theorem 1.1] relating the Brown-Kervaire invariant and the signature modulo 8 is formulated geometrically as follows:

Theorem 2.2. *([Mor71, theorem 1.1]) Let* X *be a* $4k$*-dimensional Poincaré complex, then*

$$\sigma(X) \equiv \mathrm{BK}(H^{2k}(X;\mathbb{Z}_2), \mathcal{P}_2) \pmod 8,$$

where $\mathcal{P}_2$ *is the Pontryagin square.*

Making use of the theory about the Brown-Kervaire invariant presented in [BR06], I have provided a new simpler proof of this Theorem by Morita by using chain complexes.

2.2 Multiplicativity modulo 8

In my thesis I have also proved the following result:

Theorem 2.3. *Let* $F \to E \to B$ *be an oriented fibre bundle of closed connected Poincaré complexes such that the action of* $\pi_1(B)$ *on* $H^{2m+1}(F,\mathbb{Z}_2)/torsion \otimes Z_2$ *is trivial, then*

$$\sigma(E) \equiv \sigma(F)\sigma(B) \pmod 8.$$

To prove this Theorem, I have first stated its algebraic analog. Using the idea of transfer maps in L-theory from [LR88] an appropriate algebraic model for the total space of the fibre bundle can be described. Two important tools for this proof are Theorem 2.2 of Morita and the expression of the Pontryagin square of the total space in terms of the equivariant Pontryagin square of the base and the intersection form of the fibre. The equivariant Pontryagin square was defined in [Kor05] and I have extended this definition for odd-dimensional cohomology classes.

The proof that I have given for Theorem 2.3 includes the case when the base and fibre are of dimensions $4m+2$ and $4n+2$ respectively. In this case the theorem takes the following form:

Corollary 2.4. *Let $F^{4m+2} \to E^{4k} \to B^{4n+2}$ be an oriented fibre bundle such that the action of $\pi_1(B)$ on $H^{2m+1}(F, \mathbb{Z}_2)/torsion \otimes Z_2$ is trivial, then*

$$\sigma(E) \equiv 0 \pmod 8.$$

2.3 Examples and computations

One major problem in the context of Theorem 2.3 has been to find non-trivial examples that satisfy this Theorem. The surface bundle examples of Atiyah and Kodaira have signature equal to 8 or a multiple of 8. To my knowledge, the only example of a surface bundle with signature 4 in the literature was given by Endo in [End98]. Endo used Meyer's arguments to construct a surface bundle which has as basis an orientable surface of genus 111 and as fibre an orientable surface of genus 3. The action of the fundamental group is not given explicitly in the paper. I have written down the action explicitly to confirm that in this example the action of $\pi_1(B)$ on $H^1(F, \mathbb{Z}_2)$ is non-trivial, as expected from Theorem 2.3.

3 Research Objectives and Methods

Project 1: Relation between the signature of a fibre bundle modulo 8 and other invariants

In the context of this project I am currently interested in the aspects described in **A**, **B** and **C** below.

1. A. Relation of the signature mod 8 and the Arf invariant

By [Mor71, Theorem 1.1], (see 2.2) we know that the signature of a $4k$-dimensional Poincaré complex is congruent modulo 8 to the $\mathbb{Z}_8$-valued Brown-Kervaire invariant of a Pontryagin square.

Meyer [Mey73, Theorem 3] proved that the signature of the total space of a surface bundle is divisible by 4. This result was later generalized to high-dimensions by Hambleton, Korzeniewski and Ranicki [HKR07, Theorem A]. Combining these Theorems and Theorem 2.2 of Morita, we deduce that if X in the Theorem of Morita is the total space of a surface bundle, then the Brown-Kervaire invariant will only attain two values in $\mathbb{Z}_8$, it can be either 0 or 4:

Theorem 3.1. *Let $F^{4m+2} \to E^{4k} \to B^{4n+2}$ be an oriented fibre bundle, then*

$$\sigma(E) = \mathrm{BK}(H^{2k}(E; \mathbb{Z}_2), \mathcal{P}_2) = 0 \text{ or } 4 \pmod 8.$$

Furthermore, it was proven by Brown in [Bro72, Theorem 1.20] that the Brown-Kervaire invariant satisfies the following relationship with the classical Arf invariant, namely:

Theorem 3.2. *([Bro72, Theorem 1.20]) If $h : V \to \mathbb{Z}_4$ is a quadratic form and it can be expressed as a multiple of another quadratic form, $h = jh'$ with $j : \mathbb{Z}_2 \to \mathbb{Z}_4$ the non-trivial homomorphism, then*

$$BK(V, h) = 4\mathrm{Arf}(V, h') \in \mathbb{Z}_8.$$

I am interested in investigating if the signature in Theorem 3.1 can be expressed in terms of the classical $\mathbb{Z}_2$-valued Arf invariant.

Conjecture 3.3. *Let $F^{4m+2} \to E^{4k} \to B^{4n+2}$ be an oriented fibre bundle, then*

$$\sigma(E) = 4\mathrm{Arf}(H^{2k}(E;\mathbb{Z}_2), \phi_1) \pmod 8,$$

where $\phi_1(u,v) \in \mathbb{Z}_2$ for all $(u,v) \in H^{2k}(E;\mathbb{Z}_2)$ is defined as the cup-1 product.

This conjecture would imply that the Pontryagin square $\mathcal{P}_2 \in \mathbb{Z}_4$ factors through $\mathbb{Z}_2$ for $F^{4m+2} \to E^{4k} \to B^{4n+2}$.

One possible approach to proving this conjecture is to compare the Brown-Kervaire invariant of the Pontryagin square and the Brown-Kervaire invariant of $2\phi_1$. That is, I would like to prove the following relation

$$\mathrm{BK}(H^{2k}(E;\mathbb{Z}_2), \mathcal{P}_2) - \mathrm{BK}(H^{2k}(E;\mathbb{Z}_2), 2\phi_1) = 0 \in \mathbb{Z}_8. \tag{1}$$

When the intersection form is even, then it is clear that the Pontryagin square

$$P_2(u,v) = \phi_0(v,v) + 2\phi_1(u,v) \in \mathbb{Z}_4$$

evaluated on any element $(u,v) \in H^{2k}(E;\mathbb{Z}_2)$ is divisible by 2. In this case it is straightforward to see that the conjecture holds. It is more difficult to prove the equality in equation (1) when the intersection form is not always even. One possible strategy is to use the definition of the Brown-Kervaire invariant as a Gauss sum and study what are the possible values for both Brown-Kervaire invariants in equation (1). Alternatively we can take the following approach:

(a) Prove that the difference between the Brown-Kervaire invariants is given by the Pontryagin square on a characteristic element, like the Wu class $v_{2k}(\nu_E) \in H^{2k}(E;\mathbb{Z}_2)$ of the normal bundle of the total space E,

$$\mathrm{BK}(H^{2k}(E;\mathbb{Z}_2), \mathcal{P}_2) - \mathrm{BK}(H^{2k}(E;\mathbb{Z}_2), 2\phi_1) = \mathcal{P}_2(v_{2k}(\nu_E)) \in \mathbb{Z}_4.$$

(b) Prove that for a fibre bundle with $\sigma(E) \equiv 0 \pmod 4$, it holds that

$$\mathcal{P}_2(v_{2k}(\nu_E)) = 0.$$

If Conjecture 3.3 is true, then we obtain an interesting interpretation of the non multiplicativity modulo 8 of the signature. Since the Arf invariant with value 1 implies a non-trivial action of the fundamental group on the middle cohomology of the fibre with $\mathbb{Z}_2$ coefficients, while a trivial action implies that the Arf invariant takes value 0.

1. B. Relation between the signature modulo 8 and the mod 8 signature in the context of fibrations

A $4k$-dimensional normal complex X has two basic homotopy invariants: the signature $\sigma(X) \in \mathbb{Z}$ and the mod 8 signature $\widehat{\sigma}(X) \in \mathbb{Z}_8$. When the space X is Poincaré the mod 8 signature $\widehat{\sigma}(X) \in \mathbb{Z}_8$ is just the modulo 8 reduction of the signature $\sigma(X) \in \mathbb{Z}$. An important fact is that the mod 8 signature $\widehat{\sigma}(X) \in \mathbb{Z}_8$ is not a cobordism invariant, while the ordinary signature is. Ranicki and Taylor [RT] discuss the fine relation between these two invariants and provide formulae for the mod 8 signature. I am interested in the application of these ideas in the context of fibre bundles:

(a) To what extent do the theorems about the signature modulo 8 from my thesis extend to results involving the mod 8 signature.

(b) Under what conditions is the mod 8 signature of a fibration of normal complexes multiplicative.

1. C. Relation between the signature of a surface bundle and the Euler characteristic

Hamenstädt [Ham12] describes a bound of the signature by the Euler characteristic for $E \to B$ an aspherical surface bundle over a surface. Using a variant of the Milnor Wood inequality she shows that $3|\sigma(E)| \leq \chi(E)$.

I am interested in learning about these bounds, as this approach may provide a useful tool in constructing new examples of surface bundles with non-trivial signature.

A different approach to constructing such non-trivial examples is discussed in Project 2 below.

Project 2: Mumford's conjecture and the non-multiplicativity of the signature in high dimensions

One major problem in the context of the non-multiplicativity of the signature is to find non-trivial examples of bundles with non-zero signatures. Atiyah, Kodaira and Hirzebruch gave examples of oriented surface bundles $F^2 \to E^4 \to B^2$ that have signature equal to 8 or a multiple of 8. To my knowledge the only example of a surface bundle with signature 4 in the literature was given by Endo in [End98]. Endo used Meyer's arguments in [Mey73] to construct a surface bundle which has as basis an orientable surface of genus 111 and as fibre an orientable surface of genus 3.

Smooth fibre bundles with fibre F are classified by the classifying space of the orientation preserving diffeomorphism group $B\mathrm{Diff}(F)$, in the sense that for a smooth manifold B there is a natural bijection between the set of isomorphism classes of smooth fibre bundles $F \to E \to B$ and the set $[B, B\mathrm{Diff}(F)]$ of homotopy classes of maps. The cohomology groups $H^*(B\mathrm{Diff}(F))$ give characteristic classes of such bundles. Understanding these groups depends on the dimension of F. In the case when F is an oriented surface, Mumford formulated a conjecture by which

$$\varprojlim H^*(B\mathrm{Diff}(F^2, D^2); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots]$$

for certain characteristic classes $\kappa_i \in H^{2i}(B\mathrm{Diff}(F^2, D^2))$. This was later proved by Madsen and Weiss in [MW07].

In [Ebe08] the divisibility of the signature is considered by studying the divisibility properties of the κ_i characteristic classes. The bundles considered in [Ebe08] are oriented bundles $\pi : E \to B$ with fibre a connected oriented closed surface F and structure group $\mathrm{Diff}^+(F)$, the group of all orientation-preserving diffeomorphisms of F.

In my current research I have created a computer program using Python in order to calculate the signature of the total space of a surface bundle by giving the explicit action of the fundamental group as input. These examples are also of surface bundles and I am interested in formulating examples for high-dimensional cases. To my knowledge such examples have not been described in the Literature.

In order to find high-dimensional examples of bundles with non-multiplicative signature, I am interested in giving an analog of the study in [Ebe08] extending his work about the divisibility of characteristic classes to the case when the fibre has dimension greater than 2. The proof of the higher dimensional analog of the Madsen-Weiss Theorem given by Galatius and Randal-Williams in [GRW14] should make it possible to construct the high-dimensional analog of the work in [Ebe08].

Project 3: Non-multiplicativity of the signature and TQFTs

As is explained in [Ban13], Novikov pointed out that the additivity property of the signature is equivalent to building a non-trivial topological quantum field theory.

Suppose that M is a closed oriented manifold of dimension $4k$ and that there are submanifolds M_1 and M_2 with common boundary N such that $M = M_1 \cup_N M_2$. That is, M is obtained by gluing along N the manifold M_1 with outgoing boundary N to the manifold M_2 with incoming boundary also N. The orientation of M restricts to the orientations of M_1 and M_2. In this situation Novikov additivity holds and the signature is additive in the following sense,

$$\sigma(M) = \sigma(M_1) + \sigma(M_2) \in \mathbb{Z}.$$

It is important here that M is a closed manifold. When M is not closed and the gluing is done over a common submanifold of N, then the signature is not additive. The failure of additivity is given by a formula of Wall which contains a Maslov triple index correction term.

The operation of cutting and pasting a closed manifold along a framed codimension 1 submanifold and pasting back the resulting boundaries by a diffeomorphism (different from the identity) was studied in [KKNO73]. This cutting and pasting operation gives rise to the so called SK-groups, which have an interesting connection with the multiplicativity of the signature of fibre bundles (See [Neu75]). It has been pointed out by Kreck and Teichner that the SK and SKK invariants described in [KKNO73] are TQFT invariants. I am interested in giving a precise description of these invariants.

Another interesting connection between TQFTs and the signature of fibre bundles has been described by Banagl [Ban13]. Banagl also points out that due to the fact that the signature of a fibre bundle $F \to E \to B$ of closed, oriented manifolds is in general not multiplicative, the action exponential $\mathbb{T}$ is a non-trivial invariant and it is generally hard to compute. I am interested in developing better ways of understanding this invariant by applying the results about the non-multiplicativity of the signature included in my PhD thesis.

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