GAME THEORY: AN INTRODUCTION–ERRATA

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Please notify me at ebarron@luc.edu for any errors. I am grateful to all of those mentioned below who notified me of the errors mentioned.

The following list of errors is current as of January 23, 2009.

(1) Stephen Conwill found the following errors.
   (a) p.8 In the table at the bottom of the page II3 should be the strategy: *If I1, then S; If I2, then S.* The strategy II4 should be: *If I1, then S; If I2, then P.*
   (b) p. 9 line 5 from the top "pass as well" should be *spin.*

(2) Dinesh Ayyappan found the following error.
   (a) p.11 line 5 from top, the word "largest" should be replaced by the word "smallest".

(3) p. 12 Lemma 1.1.3, second line of proof should be

$$v^+ = \min_j \max_i a_{i,j} \leq \max_i a_{i,j^*} \leq a_{i^*,j^*} \leq \min_j a_{i^*,j} \leq \max_i \min_j a_{i,j} = v^-.$$

p. 12 in proof of Lemma 1.1.3,"Let i^* be such that . . . $j = 1, 2, m$. Should be: "Let j^* be such that $v^+ = \max_i a_{i,j^*}$ and i^* such that $v^- = \min_j a_{i^*,j}$. Then

$$a_{i^*,j} \geq v^- = v^+ \geq a_{i,j^*}, \text{ for any } i = 1, 2, , n, j = 1, 2, , m.$$

(4) p. 16, line 6, $v^+ = \min_{x \in C} \max_{y \in D} f(x, y)$, and $v^- = \max_{y \in D} \min_{x \in C} f(x, y)$, should be

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y), \text{ and } v^- = \max_{x \in C} \min_{y \in D} f(x, y).$$

(5) p. 22, The last line of the third paragraph "These probability vectors are called mixed strategies, and will turn out to be the class correct class of strategies for each of the players." should be "These probability vectors are called mixed strategies, and will turn out to be the correct class of strategies for each of the players."

(6) p. 31, last line, remove ")."

Date: January 24, 2009.

(7) p. 34, line 3 from top, "...property 3,..." should be "...properties 3 and 5..."

(8) The following errors were found by Yan Jin .
  (a) p. 43, line 12 from bottom, $E(4,Y) = -5y + 6(1-y)$ should be $E(4,Y) = 7y - 8(1-y)$.
  (b) p. 44, line 1 from top, $E(1,X)$ should be corrected as $E(X,1)$. Line 2 from top, $E(4,X)$ should be $E(X,2)$, and $(x = 5/6, 1/3)$ should be corrected as $(x = \frac{5}{6}, v = \frac{1}{3})$.

(9) p. 47, Problem 1.29, part (a) should have $\min_j E(X,j) = -\frac{42}{9}$.

(10) p. 55, Quotation added

(11) The following errors were also found by Yan Jin
  (a) p. 68, the second line of the proof of Theorem 2.3.1 should read

$$E(X,X) = XAX^T = -XA^TX^T = -(XA^TX^T)^T = -XAX^T = -E(X,X).$$

  In other words, the third A should be A^T.
  (b) p. 69, the third line from the bottom, $(a\lambda, -b\lambda, c\lambda)$ should be $(c\lambda, -b\lambda, a\lambda)$.

(12) p. 111, line 7 from top E_2 should be E_{II}.

(13) Joe Condon found the following errors.
  (a) p.117 line 1, $v(B^T)) = \frac{1}{4}$ should be $v(B^T) = \frac{3}{4}$. In line 8 from the top, $X^{B^T} = (\frac{1}{4}, \frac{3}{4})$ should be $X^{B^T} = (\frac{3}{4}, \frac{1}{4})$. Line 9 from the top "value of $\frac{1}{4}$" should be "value of $\frac{3}{4}$."
  (b) p.123, The case $R < 0$ should have the possible solutions

$$\text{if } y = 0 \implies 1 \geq x \geq \frac{r}{R},$$
$$\text{if } 0 < y < 1 \implies x = \frac{r}{R},$$
$$\text{if } y = 1 \implies 0 \leq x \leq \frac{r}{R}.$$

  In addition, in line 2 from the bottom $R < 0$ should be $R > 0$, the figure 3.2 should have $R > 0$, and line 2 above the figure should have $R > 0$.

(14) p. 125, line 9 from bottom, $E(1,Y)$ should be $E_{\text{I}}(1,Y)$.

(15) p. 145, line 5 from bottom, Y^*T should be Y^{*T}.

(16) p. 154, problem 3.23 has the answer fixed on p. 393: should have $f(x,y,p,q) = 7x + 7y - 6xy - 6 - p - q$, and $2 - x \leq q$ should be $2x - 1 \leq q$.

(17) p. 185, Problem 4.6 : Should be: Suppose that two firms have constant unit costs $c_1 = 2, c_2 = 1$, and $\Gamma = 19$ in the Stackelberg model.
(18) p. 221, Example 5.1(4): "but will take \$1 million ...," should be "but will take \$100 million ...."
(19) p. 240, Problem 5.2(b) solution $\frac{38}{5}$ should be $\frac{22}{5}$.
(20) p. 241, Problem 5.10: $x - 2$ should be x_2.
(21) p. 246,

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x1+x2+x3=\frac{5}{2}
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should be $x1 + x2 + x3 = 5/2$.
(22) p. 265 Last paragraph before Example 5.14 should have a last sentence: At the end of this chapter you can find the Maple code to find the Shapley value.
(23) p. 306 The Maple code for the calculation of the Shapley value is added.
(24) p. 393 Problem 3.24, $Y_1 = (\frac{5}{13}, \frac{5}{13}, \frac{2}{13})$ should be $Y_1 = (\frac{6}{13}, \frac{5}{13}, \frac{2}{13})$.
(25) p. 394 Problem 3.27(c), $Y_1 = (\frac{5}{13}, \frac{5}{13}, \frac{2}{13})$ should be $Y_1 = (\frac{6}{13}, \frac{5}{13}, \frac{2}{13})$.
(26) p. 400 Problem 5.13 should have $16 - x_1 - x_2$, not $16 - x_1 - x - 2$.
(27) p. 401 Problem 5.19 solution in (b) should have $x_4 = \frac{3}{2}$, not 32.
(28) p. 404-405, Problem 6.5 should have solutions (b) and (c) switched.

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tree, which is nothing more than a picture of what happens at each stage of the game where a decision has to be made.

The numbers at the end of the branches are the payoffs to player I. The number $\frac{1}{2}$, for example, means that the net gain to player I is $500 because player II had to pay $1000 for the ability to pass and they split the pot in this case. The circled nodes are spots at which the next node is decided by chance. You could even consider Nature as another player. We analyze the game by first converting the tree to a game matrix which, in this example becomes

I/II	II1	II2	II3	II4
I1	$\frac{1}{4}$	$\frac{1}{4}$	$-\frac{1}{36}$	$-\frac{1}{36}$
I2	$-\frac{3}{2}$	0	$-\frac{3}{2}$	0

To see how the numbers in the matrix are obtained, we first need to know what the pure strategies are for each player. For player I, this is easy because she makes only one choice and that is pass (I2) or spin (I1). For player II, II1 is the strategy; if I passes, then spin, but if I spins and survives, then pass. So, the expected payoff[2] to I is

$$\text{I1 against II1}: \frac{5}{6}\left(\frac{1}{2}\right) + \frac{1}{6}(-1) = \frac{1}{4}, \quad \text{and}$$
$$\text{I2 against II1}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

Strategy II3 says the following: If I spins and survives, then spin, but if I passes, then spin and fire. The expected payoff to I is

$$\text{I1 against II3}: \frac{5}{6}\left(\frac{5}{6}(0) + \frac{1}{6}(1)\right) + \frac{1}{6}(-1) = -\frac{1}{36}, \quad \text{and}$$
$$\text{I2 against II3}: \frac{5}{6}(-2) + \frac{1}{6}(1) = -\frac{3}{2}.$$

The remaining entries are left for the reader. The pure strategies for player II are summarized in the following table.

II1	If I2, then S;	If I1, then P.
II2	If I2, then P;	If I1, then P.
II3	If I1, then S;	If I2, then S.
II4	If I1, then S;	If I2, then P.

[2]This uses the fact that if X is a random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, respectively, then $EX = \sum_{i=1}^{n} x_i p_i$. In I1 against II1, X is $\frac{1}{2}$ with probability $\frac{5}{6}$ and -1 with probability $\frac{1}{6}$. See the appendix for more.

This is actually a simple game to analyze because we see that player II will never play II1, II2, or II4 because there is always a strategy for player II in which II can do better. This is strategy II3, which stipulates that if I spins and survives the shot, then II should spin, while if I passes, then II should spin and shoot. If I passes, II gets $\frac{1}{36}$ and I loses $-\frac{1}{36}$. If I spins and shoots, then II gets $\frac{3}{2}$ and I loses $-\frac{3}{2}$. The larger of these two numbers is $-\frac{1}{36}$, and so player I should always spin and shoot. Consequently, player II will also spin and shoot.

The dotted line in Figure 1.3 indicates the optimal strategies. The key to these strategies is that no significant value is placed on surviving.

Remark. Extensive form games can take into account information that is available to a player at each decision node. This is an important generalization. Extensive form games are a topic in sequential decision theory, a second course in game theory.

Finally, we present an example in which is it clear that randomization of strategies must be included as an essential element of games.

■ EXAMPLE 1.6

Evens or Odds. In this game, each player decides to show one, two, or three fingers. If the total number of fingers shown is even, player I wins $+1$ and player II loses -1. If the total number of fingers is odd, player I loses -1, and player II wins $+1$. The strategies in this game are simple: deciding how many fingers to show. We may represent the payoff matrix as follows:

Evens	Odds		
I/II	1	2	3
1	1	-1	1
2	-1	1	-1
3	1	-1	1

The row player here and throughout this book will always want to maximize his payoff, while the column player wants to **minimize** the payoff to the row player, so that her own payoff is maximized (because it is a zero sum game). The rows are called the **pure strategies** for player I, and the columns are called the **pure strategies** for player II.

The following question arises: How should each player decide what number of fingers to show? If the row player **always** chooses the same row, say, one finger, then player II can **always** win by showing two fingers. No one would be stupid enough to play like that. So what do we do? In contrast to 2×2 Nim or Russian roulette, there is no obvious strategy that will always guarantee a win for either player.

Even in this simple game we have discovered a problem. If a player always plays the same strategy, the opposing player can win the game. It seems

For each row, find the minimum payoff in each column and write it in a new additional last column. Then the lower value is the largest number in that last column, that is, the maximum over rows of the minimum over columns. Similarly, in each column find the maximum of the payoffs (written in the last row). The upper value is the smallest of those numbers in the last row.

$$\begin{array}{cccc|l} a_{11} & a_{12} & \cdots & a_{1m} & \longrightarrow \min_j a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2m} & \longrightarrow \min_j a_{2j} \\ \vdots & \vdots & \cdots & \vdots & \\ a_{n1} & a_{n2} & \cdots & a_{nm} & \longrightarrow \min_j a_{nj} \\ \hline \downarrow & \downarrow & \cdots & \downarrow & \\ \max_i a_{i1} & \max_i a_{i2} & \cdots & \max_i a_{im} & \qquad v^- = \text{largest min} \\ & & & & \qquad v^+ = \text{smallest max} \end{array}$$

Here is the precise definition.

Definition 1.1.1 *A matrix game with matrix $A_{n\times m} = (a_{ij})$ has the lower value*

$$v^- \equiv \max_{i=1,\ldots,n} \min_{j=1,\ldots,m} a_{ij}.$$

and the upper value

$$v^+ \equiv \min_{j=1,\ldots,m} \max_{i=1,\ldots,n} a_{ij},$$

The lower value v^- is the smallest amount that player I is guaranteed to receive (v^- is player I's gain floor), and the upper value v^+ is the guaranteed greatest amount that player II can lose (v^+ is player II's loss ceiling). The **game has a value** *if $v^- = v^+$, and we write it as $v = v(A) = v^+ = v^-$. This means that the smallest max and the largest min must be equal and the row and column i^*, j^* giving the payoffs $a_{i^*,j^*} = v^+ = v^-$ are* **optimal***, or a* **saddle point in pure strategies***.*

One way to look at the value of a game is as a handicap. This means that if the value v is positive, player I should pay player II the amount v in order to make it a **fair game**, with $v = 0$. If $v < 0$, then player II should pay player I the amount $-v$ in order to even things out for player I before the game begins.

■ **EXAMPLE 1.7**

Let's work this out using 2×2 Nim.

$$\begin{array}{cccccc|cc} 1 & 1 & -1 & 1 & 1 & -1 & \longrightarrow & \min = -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & \longrightarrow & \min = -1 \\ -1 & -1 & -1 & 1 & 1 & 1 & \longrightarrow & \min = -1 \\ \hline \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \boxed{v^- = -1} \\ \max = 1 & \max = 1 & \max = -1 & \max = 1 & \max = 1 & \max = 1 & \boxed{v^+ = -1} & \end{array}$$

We see that $v^- =$ largest min $= -1$ and $v^+ =$ smallest max $= -1$. This says that $v^+ = v^- = -1$, and so 2×2 Nim has $v = -1$. The optimal strategies are located as the (row,column) where the smallest max is -1 and the largest min is also -1. This occurs at any row for player I, but player II must play column 3, so $i^* = 1, 2, 3,\ j^* = 3$. The optimal strategies are **not** at **any** row column combination giving -1 as the payoff. For instance, if II plays column 1, then II will play row 1 and receive $+1$. Column 1 is not part of an optimal strategy.

We have mentioned that the most that I can be guaranteed to win should be less than (or equal to) the most that II can be guaranteed to lose, (i.e., $v^- \le v^+$), Here is a quick verification of this fact.

For any column j we know that for any fixed row i, $\min_j a_{ij} \le a_{ij}$, and so taking the max of both sides over rows, we obtain

$$v^- = \max_i \min_j a_{ij} \le \max_i a_{ij}.$$

This is true for any column $j = 1, \ldots, m$. The left side is just a number (i.e., v^-) independent of i as well as j, and it is smaller than the right side for any j. But this means that $v^- \le \min_j \max_i a_{ij} = v^+$, and we are done.

Now here is a precise definition of a (pure) saddle point involving only the payoffs, which basically tells the players what to do in order to obtain the value of the game when $v^+ = v^-$.

Definition 1.1.2 *We call a particular row i^* and column j^* a* **saddle point in pure strategies** *of the game if*

$$a_{ij^*} \le a_{i^*j^*} \le a_{i^*j}, \text{ for all rows } i = 1, \ldots, n \text{ and columns } j = 1, \ldots, m. \tag{1.1.1}$$

Lemma 1.1.3 *A game will have a saddle point in pure strategies if and only if*

$$v^- = \max_i \min_j a_{ij} = \min_j \max_i a_{ij} = v^+. \tag{1.1.2}$$

Proof. If (1.1.1) is true, then

$$v^+ = \min_j \max_i a_{i,j} \le \max_i a_{i,j^*} \le a_{i^*,j^*} \le \min_j a_{i^*,j} \le \max_i \min_j a_{i,j} = v^-.$$

But $v^- \le v^+$ always, and so we have equality throughout and $v = v^+ = v^- = a_{i^*,j^*}$.

On the other hand, if $v^+ = v^-$ then

$$\min_j \max_i a_{i,j} = \max_i \min_j a_{i,j}.$$

Let j^* be such that $v^+ = \max_i a_{i,j^*}$ and i^* such that $v^- = \min_j a_{i^*,j}$. Then

$$a_{i^*,j} \ge v^- = v^+ \ge a_{i,j^*}, \text{ for any } i = 1, \ldots, n,\ j = 1, \ldots, m.$$

Definition 1.2.1 *Let C and D be sets. A function $f : C \times D \to \mathbb{R}$ has at least one saddle point (x^*, y^*) with $x^* \in C$ and $y^* \in D$ if*

$$f(x, y^*) \le f(x^*, y^*) \le f(x^*, y) \text{ for all } x \in C, y \in D.$$

Once again we could define the upper and lower values for the game defined using the function f, called a **continuous game**, by

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y), \quad \text{and} \quad v^- = \max_{x \in C} \min_{y \in D} f(x, y).$$

You can check as before that $v^- \le v^+$. If it turns out that $v^+ = v^-$ we say, as usual, that the **game has a value** $v = v^+ = v^-$. The next theorem, the most important in game theory and extremely useful in many branches of mathematics is called the **von Neumann minimax theorem**. It gives conditions on $f, C,$ and D so that the associated game has a value $v = v^+ = v^-$. It will be used to determine what we need to do in matrix games in order to get a value.

In order to state the theorem we need to introduce some definitions.

Definition 1.2.2 *A set $C \subset \mathbb{R}^n$ is* **convex** *if for any two points $a, b \in C$ and all scalars $\lambda \in [0, 1]$, the line segment connecting a and b is also in C, i.e., for all $a, b \in C$, $\lambda a + (1 - \lambda)b \in C, \forall\, 0 \le \lambda \le 1$.*

C is closed if it contains all limit points of sequences in C; C is bounded if it can be jammed inside a ball for some large enough radius. A closed and bounded subset of Euclidean space is compact.

A function $g : C \to \mathbb{R}$ is convex if

$$g(\lambda a + (1 - \lambda)b) \le \lambda g(a) + (1 - \lambda)g(b)$$

for any $a, b \in C, 0 \le \lambda \le 1$. This says that the line connecting $g(a)$ with $g(b)$, namely $\{\lambda g(a) + (1 - \lambda)g(b) : 0 \le \lambda \le 1\}$, must always lie above the function values $g(\lambda a + (1 - \lambda)b),\ 0 \le \lambda \le 1$.

The function is concave if $g(\lambda a + (1 - \lambda)b) \ge \lambda g(a) + (1 - \lambda)g(b)$ for any $a, b \in C, 0 \le \lambda \le 1$. A function is strictly convex or concave, if the inequalities are strict.

Figure 1.4 compares a convex set and a nonconvex set. Also, recall the common calculus test for twice differentiable functions of one variable. If $g = g(x)$ is a function of one variable and has at least two derivatives, then g is convex if $g'' \ge 0$ and g is concave if $g'' \le 0$.

Now the basic von Neumann minimax theorem.

Theorem 1.2.3 *Let $f : C \times D \to \mathbb{R}$ be a continuous function. Let $C \subset \mathbb{R}^n$ and $D \subset \mathbb{R}^m$ be convex, closed, and bounded. Suppose that $x \mapsto f(x, y)$ is concave and $y \mapsto f(x, y)$ is convex. Then*

$$v^+ = \min_{y \in D} \max_{x \in C} f(x, y) = \max_{x \in C} \min_{y \in D} f(x, y) = v^-.$$

In fact, define $y = \varphi(x)$ as the function so that $f(x, \varphi(x)) = \min_y f(x, y)$. This function is well defined and continuous by the assumptions. Also define the function $x = \psi(y)$ by $f(\psi(y), y) = \max_x f(x, y)$. The new function $g(x) = \psi(\varphi(x))$ is then a continuous function taking points in $[0, 1]$ and resulting in points in $[0, 1]$. There is a theorem, called the **Brouwer fixed-point theorem**, which now guarantees that there is a point $x^* \in [0, 1]$ so that $g(x^*) = x^*$. Set $y^* = \varphi(x^*)$. Verify that (x^*, y^*) satisfies the requirements of a saddle point for f.

1.3 MIXED STRATEGIES

Von Neumann's theorem suggests that if we expect to formulate a game model which will give us a saddle point, in some sense, we need convexity of the sets of strategies, whatever they may be, and convexity-concavity of the payoff function, whatever it may be.

Now let's review a bit. In most two-person zero sum games a saddle point in pure strategies will not exist because that would say that the players should **always** do the same thing. Such games, which include 2×2 Nim, tic-tac-toe, and many others, are not interesting when played over and over. It seems that if a player should not always play the same strategy, then there should be some randomness involved, because otherwise the opposing player will be able to figure out what the first player is doing and take advantage of it. A player who chooses a pure strategy randomly chooses a row or column according to some probability process that specifies the chance that each pure strategy will be played. These probability vectors are called **mixed strategies**, and will turn out to be the correct class of strategies for each of the players.

Definition 1.3.1 *A mixed strategy is a vector $X = (x_1, \ldots, x_n)$ for player I and $Y = (y_1, \ldots, y_m)$ for player II, where*

$$x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad y_j \geq 0, \sum_{j=1}^{m} y_j = 1.$$

The components x_i represent the probability that row i will be used by player I, so $x_i = Prob(I \text{ uses row } i)$, and y_j the probability column j will be used by player II, that is, $y_j = Prob(II \text{ uses row } j)$. Denote the set of mixed strategies with k components by

$$S_k \equiv \{(z_1, z_2, \ldots, z_k) \mid z_i \geq 0, i = 1, 2, \ldots, k, \sum_{i=1}^{k} z_i = 1\}.$$

In this terminology, a mixed strategy for player I is any element $X \in S_n$ and for player II any element $Y \in S_m$. A pure strategy $X \in S_n$ is an element of

Properties of Optimal Strategies (1.3.1)

1. If w is any number such that $E(i, Y) \leq w \leq E(X, j), i = 1, \ldots, n, j = 1, \ldots, m$, where X is a strategy for player I and Y is a strategy for player II, then $w = value(A)$ and (X, Y) must be a saddle point. This is the way to check whether you have a solution to the game. This is part (c) of Theorem 1.3.7 but worth repeating.

2. If X is a strategy for player I and $value(A) \leq E(X, j), j = 1, \ldots, n$, then X is optimal for player I. If Y is a strategy for player II and $value(A) \geq E(i, Y), i = 1, \ldots, m$, then Y is optimal for player II.

3. If Y is optimal for II and $y_j > 0$, then $E(X, j) = value(A)$ for any optimal mixed strategy X for I. Similarly, if X is optimal for I and $x_i > 0$, then $E(i, Y) = value(A)$ for any optimal Y for II. Thus, if any optimal mixed strategy for a player has a strictly positive probability of using a row or a column, then that row or column played against any optimal opponent strategy will yield the value. This result is also called the **Equilibrium Theorem**.

4. If X is any optimal strategy for player I and $E(X, j) > value(A)$ for some column j, then for any optimal strategy Y for player II, we must have $y_j = 0$. Player II would never use column j in any optimal strategy for player II. Similarly, if Y is any optimal strategy for player II and $E(i, Y) < value(A)$, then any optimal strategy X for player I must have $x_i = 0$. If row i for player I gives a payoff when played against an optimal strategy for player II strictly below the value of the game, then player I would never use that row in any optimal strategy for player I.

5. If for any optimal strategy Y for player II, $y_j = 0$, then there is an optimal strategy X for player I so that $E(X, j) > value(A)$. If for any optimal strategy X for I, $x_i = 0$, then there is an optimal strategy Y for II so that $E(i, Y) < value(A)$. This is the converse statement to property 4.

6. If player I has more than one optimal strategy, then player I's set of optimal strategies is a convex, closed, and bounded set. Also, if player II has more than one optimal strategy, then player II's set of optimal strategies is a convex, closed, and bounded set.

$x_i > 0, i = 1, 2, 3$, must be wrong. On the other hand, we know that player I has an optimal strategy of $X^* = (0, 1, 0)$, and so, by the equilibrium theorem (1.3.1), properties 3 and 5, we know that $E(2, Y) = 1$, for an optimal strategy for player II, as well as $E(1, Y) < 1$, and $E(3, Y) < 1$. We need to look for y_1, y_2, y_3 so that

$$y_1 + y_2 + y_3 = 1, \ -2y_1 + 2y_2 - y_3 < 1, 3y_1 + y_3 < 1.$$

We may replace $y_3 = 1 - y_1 - y_2$ and then get a graph of the region of points satisfying all the inequalities in (y_1, y_2) space in Figure 1.5.

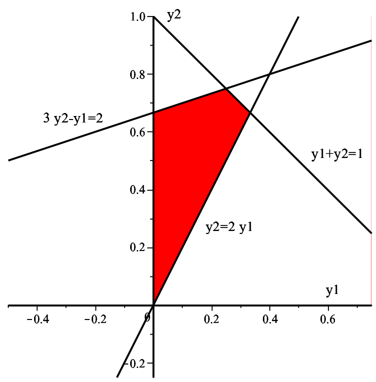


Figure 1.5 Optimal strategy set for Y.

There are lots of points which work. In particular, $Y = (0.15, 0.5, 0.35)$ will give an optimal strategy for player II in which all $y_j > 0$.

1.3.1 Dominated Strategies

Computationally, smaller game matrices are better than large matrices. Sometimes we can reduce the size of the matrix A by eliminating rows or columns (i.e., strategies) that will never be used because there is always a better row or column to use. This is elimination by **dominance.** We should check for dominance whenever we are trying to analyze a game before we begin because it can reduce the size of a matrix.

For example, if every number in row i is bigger than or equal to every corresponding number in row k, specifically $a_{ij} \geq a_{kj}, j = 1, \ldots, m$ (with strict inequality in at least one comparison), then the row player I would never play row k (since she wants the biggest possible payoff), and so we can drop it from the matrix. Similarly, if every number in column j is less than or equal to every corresponding number in column k (i.e., $a_{ij} \leq a_{ik}, i = 1, \ldots, n$), then the column player II would never play column k (since he wants player I to get the smallest possible payoff), and so we can

EXAMPLE 1.15

Let's consider

$$A = \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix}$$

This is a 4×2 game without a saddle point in pure strategies since $v^- = -1, v^+ = 6$. There is also no obvious dominance, so we try to solve the game graphically. Suppose that player II uses the strategy $Y = (y, 1-y)$, then we graph the payoffs $E(i, Y), i = 1, 2, 3, 4,$ as shown in Figure 1.10.

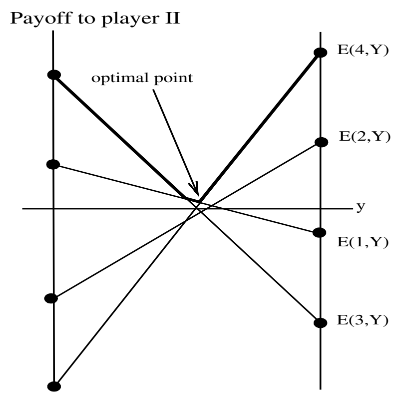


Figure 1.10 Mixed for player II versus 4 rows for player I.

You can see the difficulty with solving games graphically; you have to be very accurate with your graphs. Carefully reading the information, it appears that the optimal strategy for Y will be determined at the intersection point of $E(4, Y) = 7y - 8(1-y)$ and $E(1, Y) = -y + 2(1-y)$. This occurs at the point $y^* = \frac{5}{9}$ and the corresponding value of the game will be $v(A) = \frac{1}{3}$. The optimal strategy for player II is $Y^* = (\frac{5}{9}, \frac{4}{9})$.

Since this uses only rows 1 and 4, we may now drop rows 2 and 3 to find the optimal strategy for player I. In general, we may drop the rows (or columns) not used to get the optimal intersection point. Often that is true because the unused rows are dominated, but not always. To see that here, since $3 \leq 7\frac{1}{2} - 1\frac{1}{2}$ and $-4 \leq -8\frac{1}{2} + 2\frac{1}{2}$, we see that row 2 is dominated by a convex combination of rows 1 and 4; so row 2 may be dropped. On the other hand, there is no $\lambda \in [0, 1]$ so that $-5 \leq 7\lambda - 1(1-\lambda)$ and $6 \leq -8\lambda + 2(1-\lambda)$. Row 3 is not dominated by a convex combination of rows 1 and 4, but it is dropped because its payoff line $E(3, Y)$ does not pass through the optimal point.

Considering the matrix using only rows 1 and 4, we now calculate $E(X,1) = -x+7(1-x)$ and $E(X,2) = 2x-8(1-x)$ which intersect at $(x = \frac{5}{6}, v = \frac{1}{3})$. We obtain that row 1 should be used with probability $\frac{5}{6}$ and row 4 should be used with probability $\frac{1}{6}$, so $X^* = (\frac{5}{6}, 0, 0, \frac{1}{6})$. Again, $v(A) = \frac{1}{3}$.

A verification that these are indeed optimal uses Theorem 1.3.7(c). We check that $E(i, Y^*) \le v(A) \le E(X^*, j)$ for all rows and columns. This gives

$$\begin{bmatrix} \frac{5}{6} & 0 & 0 & \frac{1}{6} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 & 2 \\ 3 & -4 \\ -5 & 6 \\ 7 & -8 \end{bmatrix} \begin{bmatrix} \frac{5}{9} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{9} \\ -\frac{1}{9} \\ \frac{1}{3} \end{bmatrix}.$$

Everything checks.

We end this section with a simple analysis of a version of poker, at least a small part of it.

EXAMPLE 1.16

This is a modified version of the endgame in poker. Here are the rules. Player I is dealt a card that may be an ace or a king. Player I sees the result but II does not. Player I may then choose to fold or bet. If I folds, he has to pay player II \$1. If I bets, player II may choose to fold or call. If II folds, she pays player I \$1. If player II calls and the card is a king, then player I pays player II \$2, but if the card comes up ace, then player II pays player I \$2.

Why wouldn't player I immediately fold when he gets dealt a king? It is the rule that I must pay II \$1 when I gets a king and he folds. Player I is hoping that player II will fold if I bets while holding a king. This is the element of bluffing, because if II calls while I is holding a king, then I must pay II \$2. Figure 1.11 is a graphical representation of the game.

Now player I has four strategies: FF =fold on ace and fold on king, FB=fold on ace and bet on King, BF =bet on ace and fold on king, and BB =bet on ace and bet on king. Player II has only two strategies, namely, F =fold or C =call.

Assuming that the probability of being dealt a king or an ace is $\frac{1}{2}$ we may calculate the expected reward to player I and get the matrix as follows:

I/II	C	F
FF	-1	-1
FB	$-\frac{3}{2}$	0
BF	$\frac{1}{2}$	0
BB	0	1

.

1.26 Show that for any strategy $X = (x_1, \ldots, x_n) \in S_n$ and any numbers $b_1, \ldots, b_n$, it must be that

$$\max_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \max_{1 \le i \le n} b_i \text{ and } \min_{X \in S_n} \sum_{i=1}^{n} x_i b_i = \min_{1 \le i \le n} b_i.$$

1.27 The properties of optimal strategies (1.3.1) show that $X^* \in S_n$ and $Y^* \in S_m$ are optimal if and only if $\min_j E(X^*, j) = \max_i E(i, Y^*)$. The common value will be the value of the game. Verify this.

1.28 Show that if (X^*, Y^*) and (X^0, Y^0) are both saddle points for the game with matrix A, then so is (X^*, Y^0) and (X^0, Y^*). In fact, (X_λ, Y_λ) where $X_\lambda = \lambda X^* + (1-\lambda)X^0, Y_\lambda = \lambda Y^* + (1-\lambda)Y^0$ and λ is any number in $[0, 1]$.

1.29 Consider the game with matrix

$$A = \begin{bmatrix} -2 & 3 & 5 & -2 \\ 3 & -4 & 1 & -6 \\ -5 & 3 & 2 & -1 \\ -1 & -3 & 2 & 2 \end{bmatrix}.$$

Someone claims that the strategies $X^* = (\frac{1}{9}, 0, \frac{8}{9}, 0)$ and $Y^* = (0, \frac{7}{9}, \frac{2}{9}, 0)$ are optimal.

(**a**) Is that correct? Why or why not? (**Hint**: Use a previous problem.)
(**b**) If $X^* = (\frac{13}{33}, \frac{5}{33}, 0, \frac{15}{33})$ is optimal and $v(A) = -\frac{26}{33}$, find Y^*.

1.30 In the baseball game Example 1.8 it turns out that an optimal strategy for player I, the batter, is given by $X^* = (x_1, x_2, x_3) = (\frac{2}{7}, 0, \frac{5}{7})$ and the value of the game is $v = \frac{2}{7}$. It is amazing that the batter should never expect a curveball with these payoffs under this optimal strategy. What is the pitcher's optimal strategy Y^*?

1.31 In a football game we use the matrix $A = \begin{bmatrix} 3 & 6 \\ 8 & 0 \end{bmatrix}$. The first row and column represent run, and the second row and column represent pass. The offense is the row player. Column pass means defend against the pass. Use the graphical method to solve this game.

1.6 BEST RESPONSE STRATEGIES

If you are playing a game and you determine, in one way or another, that your opponent is using a particular strategy, or is assumed to use a particular strategy, then what should you do? To be specific, suppose that you are player I and you know, or simply assume, that player II is using the mixed strategy Y, optimal or not for player II. In this case you should play the mixed strategy X that maximizes $E(X, Y)$. This

CHAPTER 2

SOLUTION METHODS FOR MATRIX GAMES

I returned, and saw under the sun, that the race is not to the swift, nor the battle to the strong, ...; but time and chance happeneth to them all.

–Ecclesiates 9:11

2.1 SOLUTION OF SOME SPECIAL GAMES

Graphical methods reveal a lot about exactly how a player reasons her way to a solution, but it is not a very practical method. Now we will consider some special types of games for which we actually have a formula giving the value and the mixed strategy saddle points. Let's start with the easiest possible class of games that can always be solved explicitly and without using a graphical method.

2.1.1 2×2 Games Revisited

We have seen that any 2×2 matrix game can be solved graphically, and many times that is the fastest and best way to do it. But there are also explicit formulas giving the ball, while II anticipates where the ball will be hit. Suppose that II can return a ball hit right 90% of the time, a ball hit left 60% of the time, and a ball hit center 70% of the time. If II anticipates incorrectly, she can return the ball only 20% of the time. Score a return as $+1$ and not return as -1. Find the game matrix and the optimal strategies.

2.3 SYMMETRIC GAMES

Symmetric games are important classes of two-person games in which the players can use the exact same set of strategies and any payoff that player I can obtain using strategy X can be obtained by player II using the same strategy $Y = X$. The two players can switch roles. Such games can be quickly identified by the rule that $A = -A^T$. Any matrix satisfying this is said to be **skew symmetric**. If we want the roles of the players to be symmetric, then we need the matrix to be skew symmetric.

Why is skew symmetry the correct thing? Well, if A is the payoff matrix to player I, then the entries represent the payoffs to player I and the negative of the entries, or $-A$ represent the payoffs to player II. So player II wants to maximize the column entries in $-A$. This means that from player II's perspective, the game matrix must be $(-A)^T$ because it is always the row player by convention who is the maximizer; that is, A is the payoff matrix to player I and $-A^T$ is the payoff to player II. So, if we want the payoffs to player II to be the same as the payoffs to player I, then we must have the same game matrices for each player and so $A = -A^T$. If this is the case, the matrix must be square, $a_{ij} = -a_{ji}$, and the diagonal elements of A, namely, a_{ii}, must be 0. We can say more. In what follows it is helpful to keep in mind that for any appropriate size matrices $(AB)^T = B^T A^T$.

Theorem 2.3.1 *For any skew symmetric game $v(A) = 0$ and if X^* is optimal for player I, then it is also optimal for player II.*

Proof. Let X be any strategy for I. Then

$$E(X, X) = X\ A\ X^T = -X\ A^T\ X^T = -(X\ A^T\ X^T)^T = -X\ A\ X^T = -E(X, X).$$

Therefore $E(X, X) = 0$ and any strategy played against itself has zero payoff.

Let (X^*, Y^*) be a saddle point for the game so that $E(X, Y^*) \leq E(X^*, Y^*) \leq E(X^*, Y)$, for all strategies (X, Y). Then for any (X, Y), we have

$$E(X, Y) = X\ A\ Y^T = -X\ A^T\ Y^T = -(X\ A^T\ Y^T)^T = -Y\ A\ X^T = -E(Y, X).$$

Hence, from the saddle point definition, we obtain

$$E(X, Y^*) = -E(Y^*, X) \leq E(X^*, Y^*) = -E(Y^*, X^*) \leq E(X^*, Y) = -E(Y, X^*).$$

Then

$$-E(Y^*, X) \leq -E(Y^*, X^*) \leq -E(Y, X^*) \Longrightarrow \\ E(Y^*, X) \geq E(Y^*, X^*) \geq E(Y, X^*).$$

But this says that Y^* is optimal for player I and X^* is optimal for player II and also that $E(X^*, Y^*) = -E(Y^*, X^*) \Longrightarrow v(A) = 0$. $\square$

EXAMPLE 2.5

General Solution of 3×3 **Symmetric Games.** For any 3×3 symmetric game we must have

$$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}.$$

Any of the following conditions gives a pure saddle point:

1. $a \geq 0, b \geq 0 \Longrightarrow$ saddle at $(1,1)$ position,
2. $a \leq 0, c \geq 0 \Longrightarrow$ saddle at $(2,2)$ position,
3. $b \leq 0, c \leq 0 \Longrightarrow$ saddle at $(3,3)$ position.

Here's why. Let's assume that $a \leq 0, c \geq 0$. In this case if $b \leq 0$ we get $v^- = \max\{\min\{a,b\}, 0, -c\} = 0$ and $v^+ = \min\{\max\{-a,-b\}, 0, c\} = 0$, so there is a saddle in pure strategies at $(2,2)$. All cases are treated similarly. To have a mixed strategy, all three of these must fail.

We next solve the case $a > 0, b < 0, c > 0$ so there is no pure saddle and we look for the mixed strategies.

Let player I's optimal strategy be $X^* = (x_1, x_2, x_3)$. Then

$$\begin{aligned} E(X^*, 1) &= -ax_2 - bx_3 \geq 0 = v(A) \\ E(X^*, 2) &= ax_1 - cx_3 \geq 0 \\ E(X^*, 3) &= bx_1 + cx_2 \geq 0 \end{aligned}$$

Each one is nonnegative since $E(X^*, Y) \geq 0 = v(A)$, for all Y. Now, since $a > 0, b < 0, c > 0$ we get

$$\frac{x_3}{a} \geq \frac{x_2}{-b}, \qquad \frac{x_1}{c} \geq \frac{x_3}{a}, \qquad \frac{x_2}{-b} \geq \frac{x_1}{c}$$

so

$$\frac{x_3}{a} \geq \frac{x_2}{-b} \geq \frac{x_1}{c} \geq \frac{x_3}{a},$$

and we must have equality throughout. Thus, each fraction must be some scalar λ, and so $x_3 = a\,\lambda, x_2 = -b\,\lambda, x_1 = c\,\lambda$. Since they must sum to one, $\lambda = 1/(a - b + c)$. We have found the optimal strategies $X^* = Y^* = (c\,\lambda, -b\,\lambda, a\,\lambda)$. The value of the game, of course is zero.

For example, the matrix

$$A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 3 \\ 3 & -3 & 0 \end{bmatrix}$$

We need to define a concept of optimal play that should reduce to a saddle point in mixed strategies in the case $B = -A$. It is a fundamental and far-reaching definition due to another genius of mathematics who turned his attention to game theory in the middle twentieth century, John Nash.

Definition 3.1.1 *A pair of mixed strategies $(X^* \in S_n, Y^* \in S_m)$ is a Nash equilibrium if $E_I(X, Y^*) \leq E_I(X^*, Y^*)$ for every mixed $X \in S_n$ and $E_{II}(X^*, Y) \leq E_{II}(X^*, Y^*)$ for every mixed $Y \in S_m$. If (X^*, Y^*) is a Nash equilibrium we denote by $v_A = E_I(X^*, Y^*)$ and $v_B = E_{II}(X^*, Y^*)$ as the optimal payoff to each player. Written out with the matrices, (X^*, Y^*) is a Nash equilibrium if*

$$E_I(X^*, Y^*) = X^* A Y^{*T} \geq X A Y^{*T} = E_I(X, Y^*), \text{ for every } X \in S_n,$$

$$E_{II}(X^*, Y^*) = X^* B Y^{*T} \geq X^* B Y^T = E_{II}(X^*, Y), \text{ for every } Y \in S_m.$$

This says that neither player can gain any expected payoff if either one chooses to deviate from playing the Nash equilibrium, **assuming that the other player is implementing his or her piece of the Nash equilibrium.** On the other hand, if it is known that one player will not be using his piece of the Nash equilibrium, then the other player may be able to increase her payoff by using some strategy other than that in the Nash equilibrium. The player then uses a **best response strategy**. In fact, the definition of a Nash equilibrium says that each strategy in a Nash equilibrium is a best response strategy against the opponent's Nash strategy. Here is a precise definition for two players.

Definition 3.1.2 *A strategy $X^0 \in S_n$ is a* **best response strategy** *to a given strategy $Y^0 \in S_m$ for player II, if*

$$E_I(X^0, Y^0) = \max_{X \in S_n} E_I(X, Y^0).$$

Similarly, a strategy $Y^0 \in S_m$ is a **best response strategy** *to a given strategy $X^0 \in S_n$ for player I, if*

$$E_{II}(X^0, Y^0) = \max_{Y \in S_m} E_{II}(X^0, Y).$$

In particular, another way to define a Nash equilibrium (X^*, Y^*) is that X^* maximizes $E_{\text{I}}(X, Y^*)$ over all $X \in S_n$ and Y^* maximizes $E_{\text{II}}(X^*, Y)$ over all $Y \in S_m$. X^* is a best response to Y^* and Y^* is a best response to X^*.

If $B = -A$, a bimatrix game is a zero sum two-person game and a Nash equilibrium is the same as a saddle point in mixed strategies. It is easy to check that from the definitions because $E_{\text{I}}(X, Y) = XAY^T = -E_{\text{II}}(X, Y)$.

Note that a Nash equilibrium in pure strategies will be a row i^* and column j^* satisfying

$$a_{ij^*} \leq a_{i^*j^*} \text{ and } b_{i^*j} \leq b_{i^*j^*}, i = 1, \ldots, n, j = 1, \ldots, m.$$

Then $v(A) = \frac{3}{2}$ is the safety value for player I and $v(B^T) = \frac{3}{4}$ is the safety value for player II.

The maxmin strategy for player I is $X = (\frac{3}{4}, \frac{1}{4})$, and the implementation of this strategy guarantees that player I can get at least her safety level. In other words, if I uses $X = (\frac{3}{4}, \frac{1}{4})$, then $E_{\text{I}}(X, Y) \geq v(A) = \frac{3}{2}$ no matter what Y strategy is used by II. In fact

$$E_{\text{I}}\left(\left(\frac{3}{4}, \frac{1}{4}\right), Y\right) = \frac{3}{2}(y_1 + y_2) = \frac{3}{2}, \text{ for any strategy } Y = (y_1, y_2).$$

The maxmin strategy for player II is $Y = X^{B^T} = (\frac{3}{4}, \frac{1}{4})$, which she can use to get at least her safety value of $\frac{3}{4}$.

Is their a connection between the safety levels and a Nash equilibrium? The safety levels are the guaranteed amounts each player can get by using their own individual maxmin strategies, so any rational player must get at least the safety level in a bimatrix game. In other words, it has to be true that if (X^*, Y^*) is a Nash equilibrium for the bimatrix game (A, B), then

$$E_{\text{I}}(X^*, Y^*) = X^*AY^{*T} \geq value(A) \text{ and } E_{\text{II}}(X^*, Y^*) = X^*BY^{*T} \geq value(B^T).$$

This would say that in the bimatrix game, if players use their Nash points, they get at least their safety levels. That's what it means to be **individually rational**.

Here's why that's true.

Proof. It's really just from the definitions. The definition of Nash equilibrium says

$$E_{\text{I}}(X^*, Y^*) = X^*AY^{*T} \geq E_{\text{I}}(X, Y^*) = XAY^{*T}, \text{ for all } X \in S_n.$$

But if that is true for all mixed X, then

$$E_{\text{I}}(X^*, Y^*) \geq \max_{X \in S_n} XAY^{*T} \geq \min_{Y \in S_m} \max_{X \in S_n} XAY^T = value(A).$$

The other part of a Nash definition gives us

$$\begin{aligned} E_{\text{II}}(X^*, Y^*) = X^*BY^{*T} &\geq \max_{Y \in S_m} X^*BY^T \\ &= \max_{Y \in S_m} YB^TX^{*T} \qquad (\text{since } X^*BY^T = YB^TX^{*T}) \\ &\geq \min_{X \in S_n} \max_{Y \in S_m} YB^TX^T = value(B^T). \end{aligned}$$

Each player does at least as well as assuming the worst. □

PROBLEMS

3.1 Show that (X^*, Y^*) is a saddle point of the game with matrix A if and only if (X^*, Y^*) is a Nash equilibrium of the bimatrix game $(A, -A)$.

2. $R = 0, r > 0$. Solutions are $0 \leq x \leq 1, y = 0$.

3. $R = 0, r < 0$. Solutions are $0 \leq x \leq 1, y = 1$.

4. $R > 0$. Solutions are

$$\begin{aligned} &\text{if } y = 0 \implies 0 \leq x \leq \frac{r}{R}, \\ &\text{if } 0 < y < 1 \implies x = \frac{r}{R}, \\ &\text{if } y = 1 \implies 1 \geq x \geq \frac{r}{R}. \end{aligned}$$

5. $R < 0$. In this final case the set of all possible solutions are

$$\begin{aligned} &\text{if } y = 0 \implies 1 \geq x \geq \frac{r}{R}, \\ &\text{if } 0 < y < 1 \implies x = \frac{r}{R}, \\ &\text{if } y = 1 \implies 0 \leq x \leq \frac{r}{R}. \end{aligned}$$

We may also draw a zigzag line for player II that will be a graph of the rational reaction set for player II to a given strategy X for player I. For example, if $R > 0$, the graph would look like Figure 3.2. This bold line would be a graph of the rational

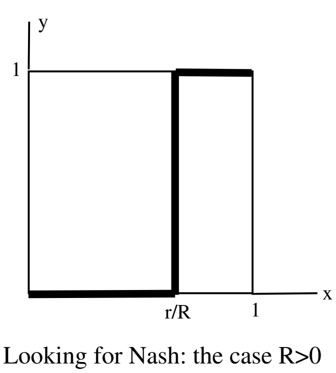


Figure 3.2 Rational reaction set for player II.

reaction set for player II against a given X. It has the explicit representation in the case $R > 0$ given by

$$R_{\text{II}} = \left\{(x, 0) \mid 0 \leq x \leq \frac{r}{R}\right\} \cup \left\{\left(\frac{r}{R}, y\right) \mid 0 < y < 1\right\} \cup \left\{(x, 1) \mid \frac{r}{R} \leq x \leq 1\right\}.$$

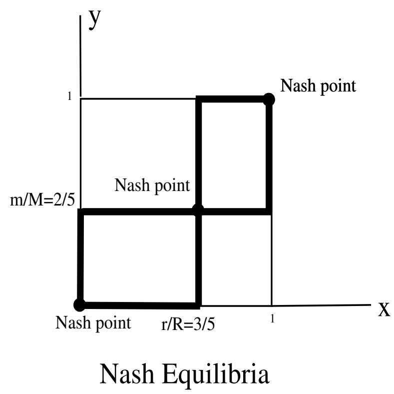


Figure 3.3 Rational reaction sets for both players

This is curious because the expected payoffs to each player are **much less** than they could get at the other Nash points.

We will see pictures like Figure 3.3 again in the next section when we consider an easier way to get Nash equilibria.

Remark: A direct way to calculate the rational reaction sets for 2×2 **games.** This is a straightforward derivation of the rational reaction sets for the bimatrix game with matrices (A, B). Let $X = (x, 1-x),\ Y = (y, 1-y)$ be any strategies and define

$$f(x,y) = E_{\mathrm{I}}(X,Y) \text{ and } g(x,y) = E_{\mathrm{II}}(X,Y).$$

The idea is to find for a fixed $0 \leq y \leq 1$, the best response to y. Accordingly,

$$\begin{aligned}\max_{0\leq x\leq 1} f(x,y) &= \max_{0\leq x\leq 1} xE_{\mathrm{I}}(1,Y) + (1-x)E_{\mathrm{I}}(2,Y)\\ &= x[E_{\mathrm{I}}(1,Y) - E_{\mathrm{I}}(2,Y)] + E_{\mathrm{I}}(2,Y)\\ &= \begin{cases} E_{\mathrm{I}}(2,Y) & \text{at } x=0 \text{ if } E_{\mathrm{I}}(1,Y) < E_{\mathrm{I}}(2,Y);\\ E_{\mathrm{I}}(1,Y) & \text{at } x=1 \text{ if } E_{\mathrm{I}}(1,Y) > E_{\mathrm{I}}(2,Y);\\ E_{\mathrm{I}}(2,Y) & \text{at any } 0<x<1 \text{ if } E_{\mathrm{I}}(1,Y) = E_{\mathrm{I}}(2,Y).\end{cases}\end{aligned}$$

Now we have to consider the inequalities in the conditions. For example,

$$E_{\mathrm{I}}(1,Y) < E_{\mathrm{I}}(2,Y) \Leftrightarrow My < m,\ M = a_{11} - a_{12} - a_{21} + a_{22},\ m = a_{22} - a_{12}.$$

If $M > 0$ this is equivalent to the condition $0 \leq y < m/M$. Consequently, in the case $M > 0$, the best response to any $0 \leq y < M/m$ is $x = 0$. All remaining cases because $XJ_n^T = J_m Y^T = 1$. But this is exactly what it means to be a Nash point. This means that (X^*, Y^*) is a Nash point if and only if

$$X^*AY^{*T}J_n^T \geq AY^{*T}, \quad (X^*BY^{*T})J_m \geq X^*B.$$

We have already seen this in Proposition 3.2.3.

Now suppose that (X^*, Y^*) is a Nash point. We will see that if we choose the scalars

$$p^* = E_{\mathrm{I}}(X^*, Y^*) = X^*AY^{*T} \text{ and } q^* = E_{\mathrm{II}}(X^*, Y^*) = X^*BY^{*T},$$

then (X^*, Y^*, p^*, q^*) is a solution of the nonlinear program. To see this, we first show that all the constraints are satisfied. In fact, by the equivalent characterization of a Nash point we just derived, we get

$$X^*AY^{*T}J_n^T = p^*J_n^T \geq AY^{*T} \quad \text{and} \quad (X^*BY^{*T})J_m = q^*J_m \geq X^*B.$$

The rest of the constraints are satisfied because $X^* \in S_n$ and $Y^* \in S_m$. In the language of nonlinear programming, we have shown that (X^*, Y^*, p^*, q^*) is a **feasible point**. The **feasible set** is the set of all points that satisfy the constraints in the nonlinear programming problem.

We have left to show that (X^*, Y^*, p^*, q^*) maximizes the objective function

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q$$

over the set of the possible feasible points.

Since every feasible solution (meaning it maximizes the objective over the feasible set) to the nonlinear programming problem must satisfy the constraints $AY^T \leq pJ_n^T$ and $XB \leq qJ_m$, multiply the first on the left by X and the second on the right by Y^T to get

$$XAY^T \leq pXJ_n^T = p, \qquad XBY^T \leq qJ_mY^T = q.$$

Hence, any **possible** solution gives the objective

$$f(X, Y, p, q) = XAY^T + XBY^T - p - q \leq 0.$$

So $f(X, Y, p, q) \leq 0$ for any feasible point. But with $p^* = X^*AY^{*T}, q^* = X^*BY^{*T}$, we have seen that (X^*, Y^*, p^*, q^*) is a feasible solution of the nonlinear programming problem and

$$f(X^*, Y^*, p^*, q^*) = X^*AY^{*T} + X^*BY^{*T} - p^* - q^* = 0$$

by definition of p^* and q^*. Hence this point (X^*, Y^*, p^*, q^*) both is feasible and gives the maximum objective (which we know is zero) over any possible feasible solution and so is a solution of the nonlinear programming problem. This shows that if we have a Nash point, it must solve the nonlinear programming problem.

firm 2's production cost is $C_2(q_2) = 2q_2 + 5$. Find the profit functions and the Nash equilibrium quantities of production and profits.

4.3 Compare profits in the model with uncertain costs and the standard Cournot model. Can you find a value of $0 < p < 1$ that maximizes firm 1's profits?

4.4 Suppose that we consider the Cournot model with uncertain costs but with three possible costs, $Prob(C_2 = c^i) = r_i,\ i = 1, 2, 3$, where $r_i \geq 0, r_1 + r_2 + r_3 = 1$. Solve for the optimal production quantities. Find the explicit production quantities when $r_1 = \frac{1}{2}, r_2 = \frac{1}{8}, r_3 = \frac{3}{8}, \Gamma = 100$, and $c_1 = 2, c^1 = 1, c^2 = 2, c^3 = 5$.

4.5 In the Stackelberg model compare the quantity produced, the profit, and the prices for firm 1 assuming that firm 2 did not exist so that firm 1 is a monopolist.

4.6 Suppose that two firms have constant unit costs $c_1 = 2, c_2 = 1$ and $\Gamma = 19$ in the Stackelberg model.

(**a**) What are the profit functions?

(**b**) How much should firm 2 produce as a function of q_1?

(**c**) How much should firm 1 produce? (**d**) How much, then, should firm 2 produce?

4.7 Set up and solve a Stackelberg model given three firms with constant unit costs c_1, c_2, c_3 and firm 1 announcing production quantity q_1.

4.8 In the Bertrand model show that if $c_1 = c_2 = c$, then $(p_1^*, p_2^*) = (c, c)$ is a Nash equilibrium.

4.9 Determine the entry deterrence level of production for firm 1 given $\Gamma = 100, a = 2, b = 10$. How much profit is lost by setting the price to deter a competitor?

4.10 We could make one more adjustment in the Bertrand model and see what effect it has on the model. What if we put a limit on the total quantity that a firm can produce? This limits the supply and possibly will put a floor on prices. Let $K \geq \frac{\Gamma}{2}$ denote the maximum quantity of gadgets that each firm can produce and recall that $D(p) = \Gamma - p$ is the quantity of gadgets demanded at price p. Find the profit functions for each firm.

4.11 Suppose that the demand functions in the Bertrand model are given by

$$q_1 = D_1(p_1, p_2) = (a - p_1 + bp_2)^+ \text{ and } q_2 = D_2(p_1, p_2) = (a - p_2 + bp_1)^+,$$

where $1 \geq b > 0$. This says that the quantity of gadgets sold by a firm will increase if the price set by the opposing firm is too high. Assume that both firms have a cost of production $c \leq \min\{p_1, p_2\}$.

(**a**) Show that the profit functions will be given by

$$u_i(p_1, p_2) = D_i(p_1, p_2)(p_i - c), \quad i = 1, 2.$$

Why? Well, $v(123) = d$ because the car will be sold for d, $v(1) = M$ because the car is worth M to player 1, $v(13) = d$ because player 1 will sell the car to player 3 for $d > M$, $v(12) = c$ because the car will be sold to player 2 for $c > M$, and so on. The reader can easily check that v is a characteristic function.

3. A customer wants to buy a bolt and a nut for the bolt. There are three players but player 1 owns the bolt and players 2 and 3 each own a nut. A bolt together with a nut is worth 5. We could define a characteristic function for this game as

$$v(123) = 5,\ v(12) = v(13) = 5,\ \ v(1) = v(2) = v(3) = 0, \quad \text{and} \quad v(\emptyset) = 0.$$

In contrast to the car problem $v(1) = 0$ because a bolt without a nut is worthless to player 1.

4. A small research drug company, labeled 1, has developed a drug. It does not have the resources to get FDA (Food and Drug Administration) approval or to market the drug, so it considers selling the rights to the drug to a big drug company. Drug companies 2 and 3 are interested in buying the rights but only if both companies are involved in order to spread the risks. Suppose that the research drug company wants \$1 billion, but will take \$100 million if only one of the two big drug companies are involved. The profit to a participating drug company 2 or 3 is \$5 billion, which they split. Here is a possible characteristic function with units in billions:

$$v(1) = v(2) = v(3) = 0, v(12) = 0.1, v(13) = 0.1, v(23) = 0, v(123) = 5,$$

because any coalition which doesn't include player 1 will be worth nothing.

5. A **simple game** is one in which $v(S) = 1$ or $v(S) = 0$ for all coalitions S. A coalition with $v(S) = 1$ is called a **winning coalition** and one with $v(S) = 0$ is a **losing coalition**. For example, if we take $v(S) = 1$ if $|S| > n/2$ and $v(S) = 0$ otherwise, we have a simple game that is a model of majority voting. If a coalition contains more than half of the players, it has the majority of votes and is a winning coalition.

6. In any bimatrix (A, B) nonzero sum game we may obtain a characteristic function by taking $v(1) = value(A)$, $v(2) = value(B^T)$, and $v(12) =$sum of largest payoff pair in (A, B). Checking that this is a characteristic function is skipped. The next example works one out.

■ EXAMPLE 5.2

In this example we will construct a characteristic function for a version of the prisoner's dilemma game in which we assumed that there was no cooperation. Now we will assume that the players may cooperate and negotiate. One form find the least core without normalizing.

5.7 A **constant sum game** is one in which $v(S) + v(N - S) = v(N)$ for all coalitions $S \subset N$. Show that any essential constant sum game must have empty core $C(0) = \emptyset$.

5.8 In this problem you will see why inessential games are of no interest. Show that an **inessential** game has one and only one imputation and is given by

$$\vec{x} = (x_1, \ldots, x_n) = (v(1), v(2), \ldots, v(n));$$

that is, each player is allocated exactly the benefit of the one-player coalition.

5.9 A player i is a **dummy** if $v(S) = v(S \cup i)$, for every $S \subset N$. It looks like a dummy contributes nothing. Show that if i is a dummy, $v(i) = 0$, and $\vec{x} \in C(0)$, then $x_i = 0$.

5.10 Show that a vector $\vec{x} = (x_1, x_2, \ldots, x_n)$ is an imputation if and only if there are nonnegative constants $a_i \geq 0, i = 1, 2, \ldots, n$, such that $\sum_{i=1}^{n} a_i = v(N) - \sum_{i=1}^{n} v(i)$, and $x_i = v(i) + a_i$ for each $i = 1, 2, \ldots, n$.

5.11 Let $\delta_i = v(N) - v(N - i)$. Show that $C(0) = \emptyset$ if $\sum_{i=1}^{n} \delta_i < v(N)$.

5.12 Verify the statement: $C(0) \neq \emptyset$ if and only if the linear program

$$\text{Minimize } z = x_1 + \cdots + x_n$$
$$\text{subject to } v(S) \leq \sum_{i \in S} x_i \text{ for every } S \subsetneq N$$

has a finite minimum, say z^*, and $z^* \leq v(N)$.

5.1.1 Finding the Least Core

The next theorem formalizes the idea above that when $e(S, \vec{x}) \leq 0$ for all coalitions, then the player should be happy with the imputation $\vec{x}$ and would not want to switch to another one.

One way to describe the fact that one imputation is better than another is the concept of domination.

Definition 5.1.8 *If we have two imputations $\vec{x} \in X, \vec{y} \in X$, and a nonempty coalition $S \subset N$, then $\vec{x}$* **dominates** *$\vec{y}$ (for the coalition S) if $x_i > y_i$ for all members $i \in S$, and $\vec{x}(S) = \sum_{i \in S} x_i \leq v(S)$.*

If $\vec{x}$ dominates $\vec{y}$ for the coalition S, then members of S prefer the allocation $\vec{x}$ to the allocation $\vec{y}$, because they get more $x_i > y_i$, for each $i \in S$, and the coalition S can so that $\vec{x}^*$ minimizes the maximum excess for any coalition S. When there is only one such allocation $\vec{x}^*$, it is the fair allocation. The problem is that there may be more than one element in the least core, then we still have a problem as to how to choose among them.

Remark: Maple Calculation of the Least Core. The point of calculating the ε-core is that the core is not a sufficient set to ultimately solve the problem in the case when the core $C(0)$ is (1) empty or (2) consists of more than one point. In case (2) the issue, of course, is which point should be chosen as the fair allocation. The ε-core seeks to address this issue by shrinking the core at the same rate from each side of the boundary until we reach a single point. We can use Maple to do this.

The calculation of the least core is equivalent to the linear programming problem

$$\begin{aligned}&\text{Minimize } z\\&\text{subject to}\\&v(S) - \vec{x}(S) = v(S) - \sum_{i \in S} x_i \le z, \text{ for all } S \subsetneq N.\end{aligned}$$

The characteristic function need not be normalized. So all we really need to do is to formulate the game using characteristic functions, write down the constraints, and plug them into Maple. The result will be the smallest $z = \varepsilon^1$ that makes $C(\varepsilon^1) \neq \emptyset$, as well as an imputation which provides the minimum.

For example, let's suppose we start with the characteristic function

$$v(i) = 0,\ i = 1, 2, 3,\ v(12) = 2, v(23) = 1, v(13) = 0,\ \ v(123) = \frac{5}{2}.$$

The constraint set is the ε-core

$$\begin{aligned}C(\varepsilon) &= \{\vec{x} = (x_1, x_2, x_3) \mid v(S) - x(S) \le \varepsilon, S \subsetneq N\}\\&= \{-x_i \le \varepsilon, i = 1, 2, 3, 2 - x_1 - x_2 \le \varepsilon, 1 - x_2 - x_3 \le \varepsilon,\\&\qquad 0 - x_1 - x_3 \le \varepsilon, x_1 + x_2 + x_3 = \frac{5}{2}\}\end{aligned}$$

The Maple commands used to solve this are very simple:

```
> with(simplex):
> cnsts:={-x1<=z,-x2<=z,-x3<=z,2-x1-x2<=z,1-x2-x3<=z,-x1-x3<=z,
          x1+x2+x3=5/2};
> minimize(z,cnsts);
```

Maple produces the output

$$x_1 = \frac{5}{4},\ x_2 = 1,\ x_3 = \frac{1}{4},\ z = -\frac{1}{4}.$$

formation of the grand coalition, we have

$$\underbrace{(1)(2)\cdots(|S|-2)(|S|-1)}_{|S|-1 \text{ arrive}} \quad \underbrace{(i)}_{i \text{ arrives}} \quad \underbrace{(n-|S|)(n-|S|-1)\cdots(2)(1)}_{\text{remaining arrive}}$$

Remember that because a characteristic function is superadditive, the players have the incentive to form the grand coalition.

For a given coalition S, by elementary probability, there are $(|S|-1)!(n-|S|)!$ ways i can join the grand coalition N, joining S first. With this reasoning, we assume that Z_i has the probability distribution

$$Prob(Z_i = S) = \frac{(|S|-1)!(n-|S|)!}{n!}.$$

We choose this distribution because $|S|-1$ players have joined before player i, and this can happen in $(|S|-1)!$ ways; and $n-|S|$ players join after player i, and this can happen in $(n-|S|)!$ ways. The denominator is the total number of ways that the grand coalition can form among n players. Any of the $n!$ permutations has probability $\frac{1}{n!}$ of actually being the way the players join. This distribution assumes that they are **all equally likely**. One could debate this choice of distribution, but this one certainly seems reasonable. Also, see Example 5.17 below for a direct example of the calculation of the arrival of a player to a coalition and the consequent benefits.

Therefore, for the fixed player i, the benefit player i brings to the coalition Z_i is $v(Z_i) - v(Z_i - i)$. It seems reasonable that the amount of the total grand coalition benefits that should be allocated to player i should be the expected value of $v(Z_i) - v(Z_i - i)$. This gives,

$$\begin{aligned} x_i \equiv E[v(Z_i) - v(Z_i - i)] &= \sum_{\{S \in \Pi_i\}} [v(S) - v(S-i)] Prob(Z_i = S) \\ &= \sum_{\{S \in \Pi^i\}} [v(S) - v(S-i)] \frac{(|S|-1)!(n-|S|)!}{n!}. \end{aligned}$$

The **Shapley value (or vector)** is then the allocation $\vec{x} = (x_1, \ldots, x_n)$. At the end of this chapter you can find the Maple code to find the Shapley value.

■ EXAMPLE 5.14

Two players have to divide \$M, but they each get zero if they can't reach an agreement as to how to divde it. What is the fair division? Obviously, without regard to the benefit derived from the money the allocation should be $M/2$ to each player. Let's see if Shapley gives that.

Define $v(1) = v(2) = 0, v(12) = M$. Then

$$x_1 = [v(1) - v(\emptyset)] \frac{0!1!}{1!} 2! + [v(12) - v(2)] \frac{1!0!}{2!} = \frac{M}{2}.$$

at another job. In negotiations with the union, the firm agrees to the pay level p and to employ $0 \leq w \leq W$ workers. We may consider the payoff functions as

$$u(p, w) = f(w) - pw \quad \text{to the company}$$

and

$$v(p, w) = pw + (W - w)p_0 \quad \text{to the union.}$$

Assume the safety security point is $u^* = 0$ for the company and $v^* = Wp_0$ for the union.

(**a**) What is the nonlinear program to find the Nash bargaining solution?

(**b**) Assuming an interior solution, show that the solution (p^*, w^*) of the Nash bargaining solution satisfies

$$w^* f'(w^*) = p_0 \text{ and } p^* = \frac{p_0 + f(w^*)}{2w^*}.$$

(**c**) Find the Nash bargaining solution for $f(w) = aw + b, a > 0$.

THE SHAPLEY VALUE WITH MAPLE

The following Maple commands can be used to calculate the Shapley value of a cooperative game. All you need to do is to let S be the set of numbered players, and define the characteristic function as v. The list $M = [M[k]]$ consists of all the possible coalitions.

```
>restart:with(combinat):S:={1,2,3,4};
>L:=powerset(S):M:=convert(L,list):M:=sort(M,length);K:=nops(L);
># Define the characteristic function
>for k from 1 to K do if nops(M[k])<=1 then v(M[k]):=0; end if;end do;
v({1,2}):=0:v({1,3}):=0:v({2,3}):=0:v({1,4}):=5:v({2,4}):=10:v({3,4}):=0:
v({1,2,3}):=0:v({1,3,4}):=25:v({2,3,4}):=30:v({1,2,4}):=35:v({1,2,3,4}):=55:
># Calculate Shapley
> for i from 1 to nops(S) do
  x[i]:=0:
     for k from 1 to K do
        if member(i,M[k]) and nops(M[k])>=1 then
           x[i]:=x[i]+(v(M[k])-v(M[k] minus {i}))*
                ((nops(M[k])-1)!*(nops(S)-nops(M[k]))!)/nops(S)!
        end if;
     end do;
  end do:

> for i from 1 to nops(S) do lprint(shapley[i]=x[i]); end do;
```

BIBLIOGRAPHIC NOTES

The pioneers of the theory of cooperative games include L. Shapley, W. F. Lucas, M. Shubik, and many others, but may go back to Francis Edgeworth in the 1880s. It received a huge boost in the publication in 1944 of the seminal work by von Neumann and Morgenstern [26] and then again in a 1953 paper by L. Shapley in which he introduced the Shapley value of a cooperative game.

There are many very good discussions on cooperative game theory, and they are listed in the references. The conversion of any N-person non-zero sum game to characteristic form is due to von Neumann and Morgenstern, which we follow, as presented in references by Wang [28] and Jones [7]. Example 5.9 (used here with permission of Mesterton-Gibbons) is called the "log hauling problem" by Mesterton-Gibbons [15] as a realistic example of a game with empty core. It is a good candidate to illustrate how the least core with a positive ε^1 results in a fair allocation in which all the players are dissatisfied with the allocation. The use of Maple to plot and animate $C(\varepsilon)$ as ε varies is a great way to show what is happening with the level of dissatisfaction and the resulting allocations. For the concept of the nucleolus we follow the sequence in Wang's book [28], but this is fairly standard. The allocation of costs and savings games can be found in the early collection of survey papers in reference [13]. Problem 5.19 is a modification of a scheduling problem known as the "antique dealer's problem" in Mesterton-Gibbon's fine book [15], in which we may consider savings games in **time** units rather than monetary units.

The Shapley value is popular because it is relatively easy to compute but also because, for the most part, it is based on a commonly accepted set of economic principles. The United Nations Security Council example (Example 5.19) has been widely used as an illustration of quantifying the power of members of a group. The solution given here follows the computation by Jones [7]. Example 5.20 is adapted from an example due to Aliprantis and Chakrabarti [1] and gives an efficient way to compute the Shapley allocation of expenses to multiple users of a resource, and taking into account the ability to pay and requirement to meet the expenditures.

The theory of bargaining presented in Section 5.4 has two primary developers: Nash and Shapley. Our presentation for finding the optimal threat strategies in section 5.4.2 follows that in Jones' book [7]. The alternative method of bargaining using the KS solution is from Aliprantis and Chakrabarti [1], where more examples and much more discussion can be found. Our union versus management problem (Problem 5.31) is a modification of an example due to Aliprantis and Chakrabarti [1].

We have only scratched the surface of the theory of cooperative games. Refer to the previously mentioned references and the books by Gintis [4], Rasmussen [22], and especially the book by Osborne [19], for many more examples and further study of cooperative games.

3.21

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (1,0,0) & E_{\mathrm{I}} = 2, E_{\mathrm{II}} = 1 \\ X_2 = (1,0) & Y_2 = (\frac{1}{2},0,\frac{1}{2}) & E_{\mathrm{I}} = \frac{1}{2}, E_{\mathrm{II}} = 1 \\ X_3 = (0,1) & Y_3 = (0,0,1) & E_{\mathrm{I}} = 1, E_{\mathrm{II}} = 3 \end{array}$$

3.22 Take $B = -A$. The Nash equilibrium is $X^* = (\frac{5}{8}, \frac{3}{8}, 0), Y^* = (0, \frac{5}{8}, \frac{3}{8})$, and the value of the game is $v(A) = \frac{1}{8}$.

3.23 The objective function is $f(x, y, p, q) = 7x + 7y - 6xy - 6 - p - q$ with constraints $2y - 1 \leq p, 5y - 3 \leq p$, $2x - 1 \leq q, 5x - 3 \leq q$, and $0 \leq x, y \leq 1$.

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (0,1) & E_{\mathrm{I}} = -1, E_{\mathrm{II}} = 2 \\ X_2 = (0,1) & Y_2 = (1,0) & E_{\mathrm{I}} = 2, E_{\mathrm{II}} = -1 \\ X_3 = (\frac{2}{3},\frac{1}{3}) & Y_3 = (\frac{2}{3},\frac{1}{3}) & E_{\mathrm{I}} = \frac{1}{3}, E_{\mathrm{II}} = \frac{1}{3} \end{array}$$

3.24 $X_1 = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right), Y_1 = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right), E_{\mathrm{I}} = \frac{10}{13},\ E_{\mathrm{II}} = 1$. $X_2 = \left(\frac{3}{4}, 0, \frac{1}{4}\right) = Y_2$ with payoffs $E_{\mathrm{I}} = \frac{5}{4}, E_{\mathrm{II}} = \frac{3}{2}$. $X_3 = Y_3 = (0, 1, 0)$.

3.26 The matrices are

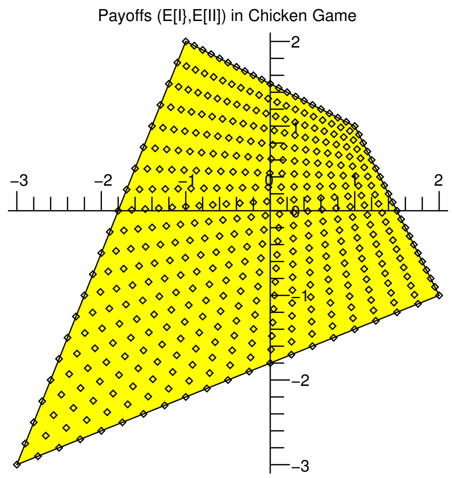
$$A = \begin{bmatrix} 1.20 & -0.56 & -0.88 & -1.2 \\ 1.24 & -0.40 & -1.44 & -1.6 \\ 0.92 & -0.04 & -1.20 & -1.8 \\ 0.6 & -0.2 & -0.6 & -2 \end{bmatrix}, B = \begin{bmatrix} 0.64 & 0.92 & 0.76 & 0.6 \\ -0.28 & 0.16 & -0.12 & -0.2 \\ -0.44 & 0.28 & 0.04 & -0.6 \\ -0.6 & 0.2 & 0.6 & 0 \end{bmatrix}.$$

One Nash equilibrium is $X = (0.71, 0, 0, 0.29), Y = (0, 0, 0.74, 0.26)$. So Pierre fires at 10 paces about 75% of the time and waits until 2 paces about 25% of the time. Bill, on the other hand, waits until 4 paces before he takes a shot but 1 out of 4 times waits until 2 paces.

3.27 (**a**) The Nash equilibria are

$$\begin{array}{lll} X_1 = (1,0) & Y_1 = (0,1) & E_{\mathrm{I}} = -1, E_{\mathrm{II}} = 2 \\ X_2 = (0,1) & Y_2 = (1,0) & E_{\mathrm{I}} = 2, E_{\mathrm{II}} = -1 \\ X_3 = \left(\frac{2}{3},\frac{1}{3}\right) & Y_3 = \left(\frac{2}{3},\frac{1}{3}\right) & E_{\mathrm{I}} = \frac{1}{3}, E_{\mathrm{II}} = \frac{1}{3} \end{array}$$

They are all Pareto-optimal because it is impossible for either player to improve their payoff without simultaneously decreasing the other player's payoff, as you can see from the figure:



None of the Nash equilibria are payoff-dominant. The mixed Nash (X_3, Y_3) risk dominates the other two.

(**b**) The Nash equilibria are

$$X_1 = \left(\frac{1}{4}, \frac{3}{4}\right), Y_1 = (1,0), E_1 = 3, E_2 = 0,$$
$$X_2 = (0,1), Y_2 = (0,1), E_1 = E_2 = 1,$$
$$X_3 = (1,0) = Y_3, E_1 = E_2 = 3.$$

(X_3, Y_3) is payoff-dominant and Pareto-optimal.

(**c**)

$$X_1 = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right), Y_1 = \left(\frac{6}{13}, \frac{5}{13}, \frac{2}{13}\right), E_{\text{I}} = \frac{10}{13}, E_{\text{II}} = 1.$$
$$X_2 = \left(\frac{3}{4}, 0, \frac{1}{4}\right) = Y_2, E_{\text{I}} = \frac{5}{4}, E_{\text{II}} = \frac{3}{2}.$$
$$X_3 = Y_3 = (0,1,0), E_{\text{I}} = 2, E_{\text{II}} = 3.$$

Clearly X_3, Y_3 is payoff-dominant and Pareto-optimal. Neither (X_1, Y_1) nor (X_2, Y_2) are Pareto-optimal relative to the other Nash equilibria, but they each risk dominate (X_3, Y_3).

(**b**) For the data given in the problem $s^* = 338.66, g^* = 357.85$.

4.23 The cumulative distribution function is $F(p) = -2p^3 + 3p^2, 0 < p < 1$. The interior solution of $1 - F(p) - pf(p) = 0$ is $p^* = 0.422$, so the reserve price should be set at 42.2% of the normalized range of prices. Notice that even though the density is symmetric around $p = \frac{1}{2}$, the optimal reserve price is not 0.5. The Maple commands to solve are

```
> restart: f:=x->6*x*(1-x);
> F:=x->int(f(y),y=0..x);
> fsolve(1-F(x)-x*f(x)=0,x);
```

4.25 The expected payoff of a bidder with valuation v who makes a bid of b is given by

$$u(b) = vProb(b \text{ is high bid}) - b = vF(\beta^{-1}(b))^{N-1} - b = v\beta^{-1}(b)^{N-1} - b.$$

Differentiate, set to zero, and solve to get $\beta(v) = ((N-1)/N)v^N$.

Since all bidders will actually pay their own bids and each bid is $\beta(v) = (N - 1/N)v^N$, the expected payment from each bidder is

$$E[\beta(V)] = \frac{N-1}{N}\int_0^1 v^N dv = \frac{N-1}{N(N+1)}.$$

Since there are N bidders, the total expected payment to the seller will be $(N-1)/(N+1)$.

SOLUTIONS FOR CHAPTER 5

5.2 (**a**) $v(1) = value(A) = \frac{8}{5}, v(2) = value(B^T) = \frac{8}{5}, v(12) = 6, v(\emptyset) = 0$.

(**b**) The core is $C(0) = \{(6 - x_2, x_2) \mid \frac{8}{5} \le x_2 \le \frac{22}{5}\}$.

(**c**) The least core is $C\left(-\frac{7}{5}\right) = \{(3,3)\}$.

5.3 In normalized form simply divide each number by 13 : $\vec{x}$ unnormalized=$\left\{\left(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2}\right)\right\}$.

5.4 $v(\emptyset) = 0, v(1) = \frac{3}{5}, v(2) = 2, v(3) = 1, v(12) = 5, v(13) = 4, v(23) = 3, v(123) = 16$.

5.5 $\varepsilon^1 = \frac{1}{3}$ and $C\left(\frac{1}{3}\right) = \left\{\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\right\}$.

5.6 $\varepsilon^1 = -1$ least core is $C(-1) = \{\vec{x} = (2,2,0)\}$.

5.7 Suppose $\vec{x} \in C(0)$ so that $e(S, \vec{x}) \le 0, \forall S \subsetneq N$. Take the single player coalition $S = \{i\}$ so $v(i) + v(N - i) = v(N)$. Since the game is essential, $v(N) > \sum_{i=1}^n v(i)$.

Since $\vec{x}$ is in the core, we have

$$v(N) > \sum_{i=1}^{n} v(i) = \sum_{i=1}^{n} v(N) - v(N-i) = nv(N) - \sum_{i=1}^{n} v(N-i),$$
$$\Longrightarrow$$
$$v(N)(n-1) < \sum_{i=1}^{n} v(N-i) \leq \sum_{i=1}^{n} \sum_{j \neq i} x_j = \sum_{i=1}^{n} v(N) - x_i$$
$$= nv(N) - \sum_{i=1}^{n} x_i = (n-1)v(N) \Rightarrow\Leftarrow .$$

5.8 Since the game is inessential, $v(N) = \sum_{i=1}^{n} v(i)$. It is obvious that $\vec{x} = (v(1), \ldots, v(n)) \in C(0)$. If there is another $\vec{y} \in C(0), \vec{y} \neq \vec{x}$, there must be one component $y_i < v(i)$ or $y_i > v(i)$. Since $\vec{y} \in C(0)$, the first possibility cannot hold and so $y_i > v(i)$. This is true at any j component of $\vec{y}$ not equal to $v(j)$. But then, adding them up gives $\sum_{i=1}^{n} y_i > \sum_{i=1}^{n} v(i) = v(N)$, which contradicts the fact that $\vec{y} \in C(0)$.

5.9 Suppose $i = 1$. Then

$$x_1 + \sum_{j \neq 1} x_j = v(N) = v(N-1) \leq \sum_{j \neq 1} x_j,$$

and so $x_1 \leq 0$. But since $-x - 1 = v(1) - x_1 \leq 0$, we have $x_1 = 0$.

5.11 Let $\vec{x} \in C(0)$. Since $v(N-1) \leq x_2 + \cdots + x_n = v(N) - x_1$, we have $x_1 \leq v(N) - v(N-1)$. In general, $x_i \leq v(N) - v(N-i), 1 \leq i \leq n$. Now add these up to get $v(N) = \sum_i x_i \leq \sum_i \delta_i < v(N)$, which says $C(0) = \emptyset$.

5.13 The core is

$$C(0) = \{(x_1, x_2, 16 - x_1 - x_2) : \frac{3}{5} \leq x_1 \leq 13, 2 \leq x_2 \leq 12, 5 \leq x_1 + x_2 \leq 15\}.$$

The least core: $\varepsilon^1 = -\frac{62}{15}, C(\varepsilon^1) = \left\{\left(\frac{71}{15}, \frac{92}{15}, \frac{77}{15}\right)\right\}$.

5.14 $q = \frac{1}{2}\left(\frac{2}{5} + \frac{3}{10} + \frac{3}{10}\right) = \frac{1}{2}$. The characteristic function is $v(i) = 0, v(12) = v(13) = v(23) = 1, v(123) = 1$.

5.15 (**b**) To see why the core is empty, show first that it must be true $x_1 + x_2 = -2$, and $x_3 + x_4 = -2$. Then, since $-1 \leq x_1 + x_2 + x_3 = -2 + x_3$, we have $x_3 \geq 1$. Similarly $x_4 \geq 1$. But then $x_3 + x_4 \geq 2$ and that is a contradiction.

(**c**) A coalition that works is $S = \{12\}$.

5.16 $X^1 = C(-\frac{1}{10}) = \{x_1 + x_2 = \frac{9}{10}, \frac{4}{10} \leq x_1, \frac{2}{10} \leq x_2\}$. The next least core is $X^2 = C(-\frac{1}{4}) = \{(\frac{11}{20}, \frac{7}{20}, \frac{2}{20})\}$.

5.17 The least core is the set $C(-1) = \{x_1 = 1, x_2 + x_3 = 11, x_2 \geq 1, x_3 \geq 2\}$. The nucleolus is the single point $\{(1, \frac{11}{2}, \frac{11}{2})\}$

5.18 For the least core $\varepsilon^1 = -\frac{1}{2}$:

$$\begin{aligned}\text{Least core} = X^1 = C(-\frac{1}{2}) = \{&x_1 + x_2 = \frac{3}{2}, x_3 + x_4 = \frac{3}{2}, x_i \geq \frac{1}{2}, i = 1,2,3,4,\\ &x_2 + x_3 \geq \frac{3}{2}, x_1 + x_4 \geq \frac{3}{2}, x_1 + x_3 \geq \frac{5}{4}, x_2 + x_4 \geq \frac{1}{2},\\ &x_1 + x_2 + x_3 \geq \frac{3}{2}, x_1 + x_2 + x_4 \geq \frac{3}{2}, x_1 + x_3 + x_4 \geq \frac{3}{2},\\ &x_2 + x_3 + x_4 \geq \frac{3}{2}, x_1 + x_2 + x_3 + x_4 = 3\}.\end{aligned}$$

Next X^2 has $\varepsilon^2 = 1$. X^3 has $\varepsilon^3 = 3$, and nucleolus=$\{(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}, \frac{3}{4})\}$.

5.19 (**a**) The characteristic function is the number of hours saved by a coalition. $v(i) = 0$, and

$$v(12) = 4, v(13) = 4, v(14) = 3, v(23) = 6, v(24) = 2, v(34) = 2,$$

$$v(123) = 10, v(124) = 7, v(134) = 7, v(234) = 8, v(1234) = 13.$$

(**b**) Nucleolus=$\{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2})\}$ with units in hours. The least core is

$$\begin{aligned}X^1 = C(-\frac{3}{2}) = \{&x_1 + x_2 + x_3 = \frac{23}{2}, x_4 = \frac{3}{2},\\ &x_1 + x_2 + x_3 + x_4 = 13, x_1 + x_2 + x_4 \geq \frac{17}{2},\\ &x_2 + x_3 + x_4 \geq \frac{19}{2}, x_1 \geq \frac{3}{2}, x_2 \geq \frac{3}{2},\\ &x_1 + x_2 \geq \frac{11}{2}, x_3 \geq \frac{3}{2}, x_1 + x_3 \geq \frac{11}{2},\\ &x_2 + x_3 \geq \frac{15}{2}, x_1 + x_4 \geq \frac{9}{2}, x_2 + x_4 \geq \frac{7}{2},\\ &x_3 + x_4 \geq \frac{7}{2}, x_1 + x_3 + x_4 \geq \frac{17}{2}\}\end{aligned}$$

The next least core, which will be the nucleolus, is $X^2 = \{(\frac{13}{4}, \frac{33}{8}, \frac{33}{8}, \frac{3}{2})\}$ with $\varepsilon^2 = 10$.

(**c**) The schedule is set up as follows: (i) Curly works from 9:00 to 11:52.5, (ii) Larry works from 11:52.5 to 1:45, (iii) Shemp works from 1:45 to 3:30, and (iv) Moe works from 3:30 to 5:00.

In order for X_3 to be an ESS, we need

$$\frac{1}{2} > \frac{1}{2} + \frac{p}{2} - 2px + 2px^2,$$

which becomes $0 > 2p(x - \frac{1}{2})^2$, for $0 < p < p_x$. This is clearly impossible, so X_3 is not an ESS.

6.2 There are three Nash equilibria $X_1 = Y_1 = (1,0)$, $X_2 = Y_2 = (0,1)$, and the mixed $X_3 = Y_3 = (\frac{2}{3}, \frac{1}{3})$. The first two are ESSs. For X_3, $u(\frac{2}{3}, \frac{2}{3}) = \frac{2}{3}$, $u(x, \frac{2}{3}) = \frac{2}{3}$. Is $u\left(\frac{2}{3}, x\right) = \frac{2}{3} > u(x,x)$? No, because $\frac{2}{3} > x^2 + 2(1-x)^2$ is false for all $0 < x < 1$.

6.3 The only symmetric (nonstrict) Nash is $(X^* = (0,1), X^*)$. Then $u(0,0) = 1$, $u(x,0) = 1$, $u(x,x) = -2x^2 + 5x + 1$, and $u(0,x) = 5x + 1$. Hence, $u(0,0) = 1 = u(x,0)$ and $u(x,x) < u(0,x)$, for any $0 < x \leq 1$. This means that $X^* = (0,1)$ is an ESS.

6.4 The Nash equilibria and their payoffs are shown in the following table; they are all symmetric.

X^*	$u(X^*, X^*)$
$(1,0,0)$	2
$(0,1,0)$	2
$(0,0,1)$	2
$(\frac{3}{4}, \frac{1}{4}, 0)$	$\frac{5}{4}$
$(\frac{1}{4}, 0, \frac{3}{4})$	$\frac{5}{4}$
$(0, \frac{3}{4}, \frac{1}{4})$	$\frac{5}{4}$
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$\frac{2}{3}$

For $X^* = (1,0,0)$ you can see this is an ESS because it is strict. Consider next $X^* = (\frac{3}{4}, \frac{1}{4}, 0)$. Since $u(Y, X^*) = \frac{5}{4}(y_1 + y_2) - y_3/2$, the set of best response strategies is $Y = (y, 1-y, 0)$. Then $u(Y,Y) = 4y^2 - 4y + 2$,and $u(X^*, Y) = -\frac{1}{4} + 2y$. Since it is **not** true that $u(Y,Y) < u(X^*, Y)$, for all best responses $Y \neq X^*$, X^* is not an ESS.

6.5 (**a**) There is a unique Nash, strict and symmetric ESS$= (0,1)$ if $a < 0, b > 0, = (1,0)$ if $b < 0, a > 0$.

(**b**) Three Nash equilibria, all symmetric, NE$= (1,0), (0,1), X, X = (b/(a+b), a/(a+b))$. Both $(1,0), (0,1)$ are strict, so $(1,0), (0,1) \in ESS$. The mixed X is not an ESS since $E(1,1) = a > ab/(a+b) = E(X,1)$ so $ESS = \{(0,1), (1,0)\}$.

(**c**) Two strict asymmetric Nash Equilibria, one symmetric Nash Equilibrium $X = (c = b/(a+b), a/(a+b))$, but now X is ESS since

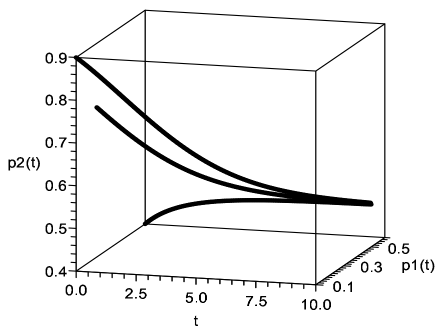
$$E(X, Y) = cay_1 + (1-c)by_2 = ab/(a+b)$$

and for every strategy $Y \neq X$, $E(Y,Y) = ay_1^2 + by_2^2 < ab/(a+b) = E(X,Y)$, so X is the ESS.

6.7 The pure Nash equilibria are clearly equivalent. For the interior mixed Nash, the calculus method shows that the partial in the appropriate variables of the payoff functions lead to equations for the Nash equilibrium independent of a, b. You may also calculate directly that $E'(X,Y) = XA'Y^T = XAY^T - (a \quad b)Y^T = E(X,Y) - (a \quad b)Y^T$. Therefore, $E'(X^*, Y^*) \geq E'(X, Y^*)$ for all X, if and only if $E(X^*, Y^*) \geq E(X, Y^*)$, for all X.

6.8 (**b**) The three Nash equilibria are $X_1 = (\frac{1}{2}, \frac{1}{2}) = Y_1$, and the two nonsymmetric Nash points $((0,1),(1,0))$ and $((1,0),(0,1))$. So only X_1 is a possible ESS.

(**c**) From the following figure you can see that $(p_1(t), p_2(t)) \to (\frac{1}{2}, \frac{1}{2})$ as $t \to \infty$ and conclude that (X_1, X_1) is an ESS. Verify directly using the stability theorem that it is asymptotically stable.



The figure shows trajectories starting from three different initial points. In the three-dimensional figure you can see that the trajectories remain in the plane $p_1 + p_2 = 1$. The Maple commands used to find the stationary solutions, check their stability, and produce the graph are

```
> restart:with(DEtools):with(plots):with(LinearAlgebra):
> A:=Matrix([[3,2],[4,1]]); X:=<x1,x2>;
> Transpose(X).A.X;
> s:=expand(%);
> L:=A.X; f:=(x1,x2)->L[1]-s;g:=(x1,x2)->L[2]-s;
> solve({f(x1,x2)=0,g(x1,x2)=0},[x1,x2]);
```