Planar Algebras and Evaluation Algorithms

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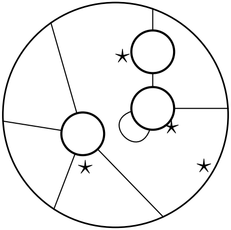
Planar algebras

Definition

A planar diagram has

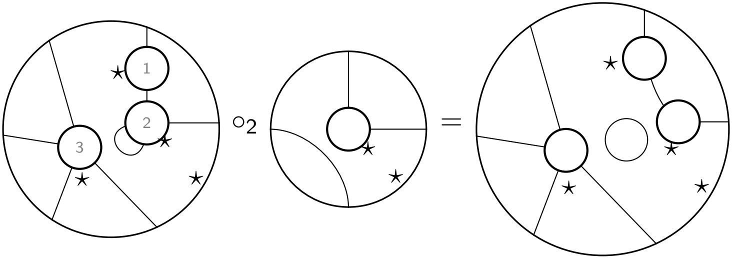
- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



In normal algebra (the kind with sets and functions), we have one dimension of composition:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

In planar algebras, we have two dimensions of composition



In abstract algebra, sets are given additional structure by functions. For example, a group is a set G with a multiplication law

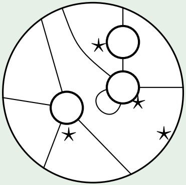
$$\circ : G \times G \to G.$$

A planar algebra also has sets, and maps giving them structure; there are a lot more of them.

Definition

A planar algebra is

- *a family of vector spaces* V_k, $k = 0, 1, 2, \ldots$, *and*
- *an interpretation of any planar diagram as a multi-linear map*

among V_i*:*  $: V_2 \times V_5 \times V_4 \to V_7$

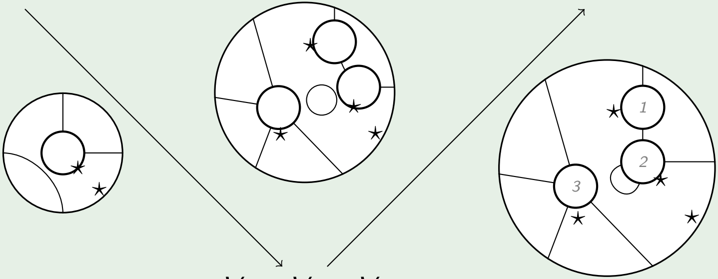
Definition

A planar algebra is

- *a family of vector spaces* V_k, $k = 0, 1, 2, \ldots$, *and*
- *Planar diagrams giving multi-linear map among* V_i,

such that composition of multilinear maps, and composition of diagrams, agree:

$$V_4 \times V_2 \times V_2 \longrightarrow V_6$$



$$V_4 \times V_4 \times V_2$$

Definition

A Temperley-Lieb diagram is a way of connecting up $2n$ *points on the boundary of a circle, so that the connecting strings don't cross.*

For example, TL_6*:* 

Example

The Temperley-Lieb planar algebra TL:

- The vector space TL_{2n} has a basis consisting of all Temperley-Lieb diagrams on $2n$ points.
- A planar diagram acts on Temperley-Lieb diagrams by placing the TL diagrams in the input disks, joining strings, and replacing closed loops of string by $\cdot\delta$.

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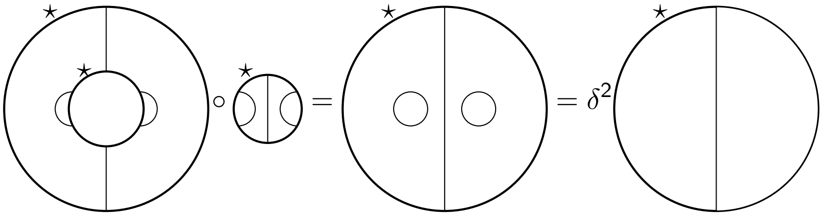


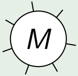
Why is Temperley-Lieb the "trivial" planar algebra? It seems to have a lot of structure.

- Any subfactor planar algebra contains a copy of Temperley-Lieb;
- As a planar algebra, Temperley-Lieb has a presentation with no generators and very few relations.

Example (Linear algebra/Matrix algebra)

$V_n = \{2^k\text{-by-}2^{n-k}$ matrices over $\mathbb{C}\}$. For now we think of a matrix as an array: a gadget that takes in indices and gives back a number.

When we insert $M \in V_n$ into a planar diagram, we think of the strings coming into it as a mechanism to specify the indices.



In more detail:
Let $V_n = \{f : \{0,1\}^n \to \mathbb{C}\}$.
If n is even, V_n is isomorphic to the set of $2^{n/2}$ by $2^{n/2}$ square matrices:

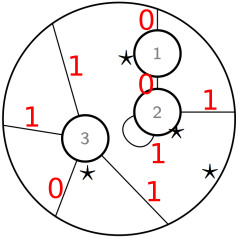
$$M \mapsto \begin{pmatrix} M(0,0,0,0) & M(0,0,1,0) & M(0,0,0,1) & M(0,0,1,1) \\ M(0,1,0,0) & M(0,1,1,0) & M(0,1,0,1) & M(0,1,1,1) \\ M(1,0,0,0) & M(1,0,1,0) & M(1,0,0,1) & M(1,0,1,1) \\ M(1,1,0,0) & M(1,1,1,0) & M(1,1,0,1) & M(1,1,1,1) \end{pmatrix}$$

How does a planar diagram act on $M \in V_n$?

Definition

A state σ on a diagram D is an assignment of 0s and 1s to strings

For example,



Definition

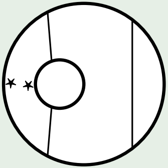
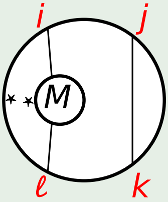
A state σ restricted to a boundary circle is the sequence of 0s and 1s you get by reading clockwise around that circle from the $\star$.

For example, $\sigma|_1 = (0,0)$, $\sigma|_2 = (1,1,0,1)$, $\sigma|_3 = (0,1,1,1)$, and $\sigma|_{\text{outside}} = (1,0,1,1,0,1)$.

So: how does a diagram D act on inputs $M_1, \ldots, M_k$? Recall that the result is supposed to be another function from V_n to $\mathbb{C}$ (n is the number of output strings for D). So, for $v \in \{0,1\}^n$, define

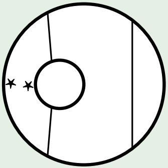
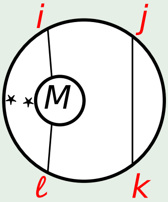
$$D(M_1, \ldots, M_k)(v) = \sum_{\substack{\text{states } \sigma \text{ s.t.} \\ \sigma|_{\text{outside}} = v}} \prod_{\text{inner disk } i} M_i(\sigma|_i)$$

Example


$(M)(i, j, k, \ell) =$


$= M(i, \ell)\delta_{j,k}.$

Example

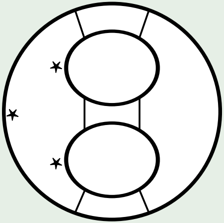
 $(M)(i, j, k, \ell) =$ 

$= M(i, \ell)\delta_{j,k}.$

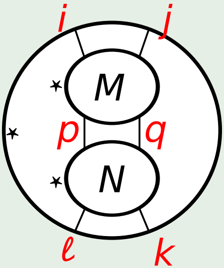
As matrices, this is an inclusion map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \hookrightarrow \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

Example

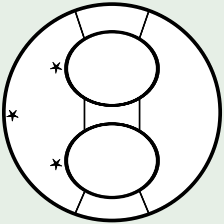


$$(M, N)(i, j, k, \ell) = \sum_{p,q \in \{0,1\}}$$



$$= \sum_{p,q \in \{0,1\}} M(i, j, q, p) N(p, q, k, \ell).$$

Example



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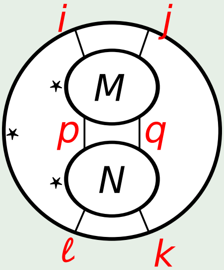


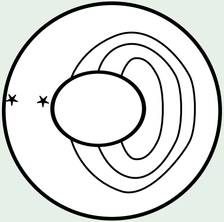
$$= \sum_{p,q\in\{0,1\}} M(i, j, q, p)N(p, q, k, \ell).$$

This is matrix multiplication!

Q: What is the multiplicative identity (as a diagram)?

Question

What are these?



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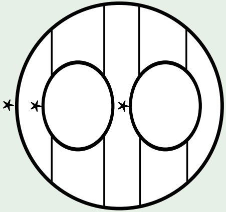

Question

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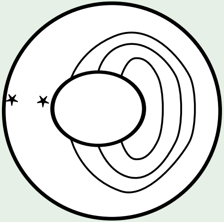

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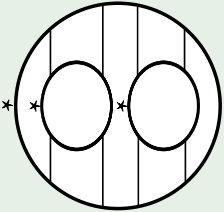


Trace tr(M) and tensor $M \otimes N$ operations.

The *n*-color theorems

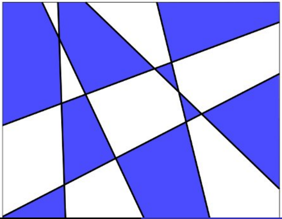
We say a graph can be <u>n-colored</u> if you can color its faces using n different colors such that adjacent regions are different colors. (To talk about "faces", the graph must be embedded on a surface. We'll stick with planar graphs.)

Definition

The <u>degree</u> of a vertex is the number of edges it has coming into it.

The two-color theorem

Any planar graph where every vertex has even degree can be two-colored.



A three-color theorem (Grötzsch 1959)

Planar graphs with no degree-three vertices can be three-colored.

The five-color theorem (Heawood 1890, based on Kempe 1879)

Any planar graph can be five-colored.

A three-color theorem (Grötzsch 1959)

Planar graphs with no degree-three vertices can be three-colored.

The five-color theorem (Heawood 1890, based on Kempe 1879)

Any planar graph can be five-colored.

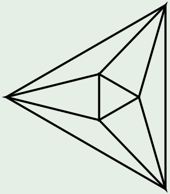
The four-color theorem (Appel-Haken 1976)

Any planar graph can be four-colored.

Definition/Theorem

The Euler characteristic of a graph is $V - E + F$. For planar graphs, $V - E + F = 2$.

Example



V=6

E=12

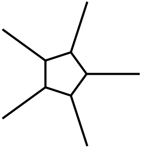
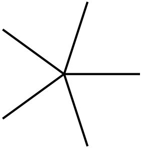
F=8

V-E+F=2

Corollary

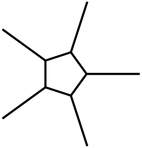
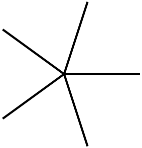
Every planar graph has a face which is either a bigon, triangle, quadrilateral or pentagon.

A new proof of the four-color theorem:

We first reduce it to a problem about graphs where every vertex has degree three. If I can color  then I can color . So, replacing every degree-n vertex with a small n-gonal face doesn't change colorability.

Thus, if a coloring theorem is true for graphs where every vertex has degree three, it is true for all graphs.

A not-so-new proof of the five-color theorem:

We first reduce it to a problem about graphs where every vertex has degree three. If I can color  then I can color . So, replacing every degree-n vertex with a small n-gonal face doesn't change colorability.

Thus, if a coloring theorem is true for graphs where every vertex has degree three, it is true for all graphs.

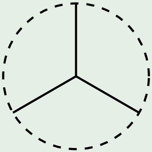
Definition

The color-counting planar algebra: The vector space V_k is functionals from length k sequences of colors to numbers:

$$V_k = \{f : \{\text{colors}\}^k \to \mathbb{N}\}$$

Any planar graph (with a boundary) is a functional from a sequence of colors, to a number: how many ways are there to color in this graph so that the boundary colors are the given sequence?

Example

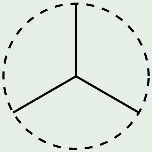


$$\{1,2,3\} \to 1$$

$$\{1,2,2\} \to 0$$

$$\{i,j,k\} \to \begin{cases} 1 & \text{if } i,j,k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

Example



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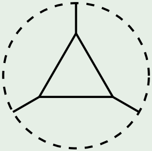
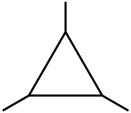
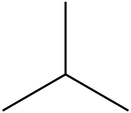
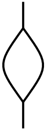
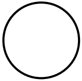
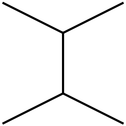
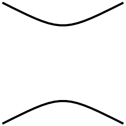
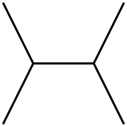
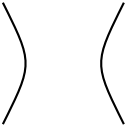


$$\{1,2,3\} \to n-3$$
$$\{1,2,2\} \to 0$$
$$\{i,j,k\} \to \begin{cases} n-3 & \text{if } i,j,k \text{ distinct;} \\ 0 & \text{else.} \end{cases}$$

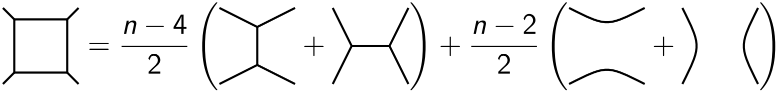
So,  $= (n-3)$  .

Similarly,  $= (n-2)$ and  $= (n-1)$.

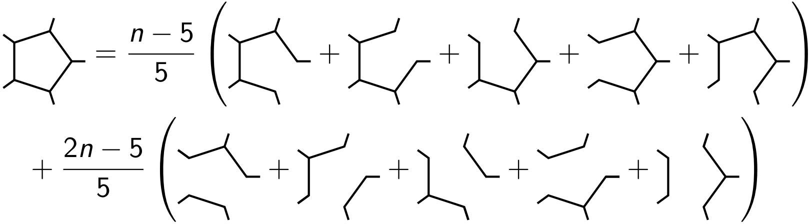
We also have a less obvious relation:

 $+$  $=$  $+$  .

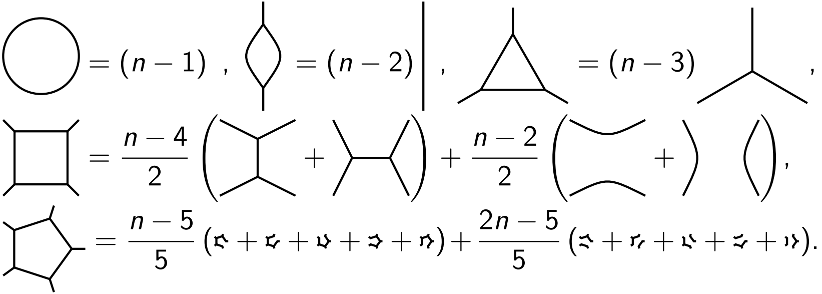
This last relation can be used to prove two more relations:



and



Proving the 5+-color theorem



All these face-removing relations are positive for $n \geq 5$.
Any planar graph contains at least one circle, bigon, triangle, quadrilateral or pentagon (via Euler characteristic). So apply one of these positive relations and repeat until you have nothing left but a sum of positive multiples of the empty diagram.

This description of how to reduce any planar graph to a multiple of the empty diagram is an example of an evaluation algorithm

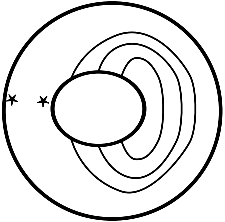
Definition

For a planar algebra with generators G and relations R, an evaluation algorithm is

- *an ordering on diagrams: When is one diagram simpler than the other?*
- *an algorithm for starting from a diagram D and repeatedly applying relations in R to get the simplest diagram equal to D.*

For example, in proving the 5+-color theorem, our ordering on planar graphs with boundary was the number of internal faces the graph had.

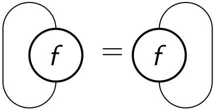
Subfactor people are interested in planar algebras where $\dim(PA_0) = 1$. Why are these easier to study?

Well, if $PA_0 \simeq \mathbb{C}$ then  $: PA_{2k} \to PA_0 \simeq \mathbb{C}$

is a trace. So we can define a bilinear form $\langle x, y \rangle := \mathrm{tr}(y^* x)$ and use geometric tools.

Subfactor planar algebras

The standard invariant of a subfactor is a planar algebra $\mathcal{P}$ with some extra structure:

- $\mathcal{P}_0$ is one-dimensional;
- All $\mathcal{P}_k$ are finite-dimensional ;
- Sphericality:  ;
- Inner product: each $\mathcal{P}_k$ has an adjoint $*$ such that $\langle x, y\rangle := \mathrm{tr}(y^* x)$ is an inner product.

Call a planar algebra with these properties a subfactor planar algebra.

Theorem (Jones, Popa)

Subfactors give subfactor planar algebras, and subfactor planar algebras give subfactors.

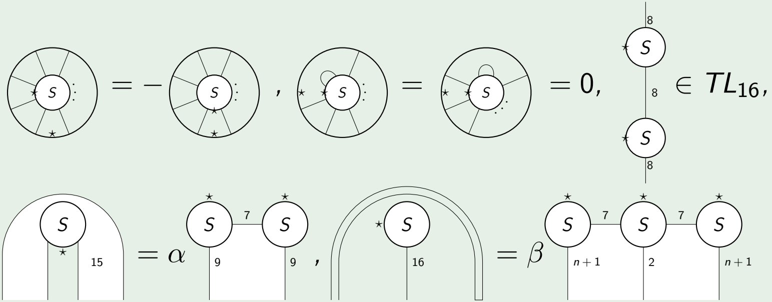
Example

Temperley-Lieb is a subfactor planar algebra if $\delta > 2$:

- TL_0 is one dimensional
- $\dim(TL_{2n}) = c_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$
- circles are circles
- Positive definiteness is the difficulty, and the only place where $\delta > 2$ comes in.

Theorem (Bigelow, Morrison, Peters, Snyder)

The extended Haagerup planar algebra $\mathcal{H}$ is the positive definite planar algebra generated by a single generator S with 16 strands, subject to the relations



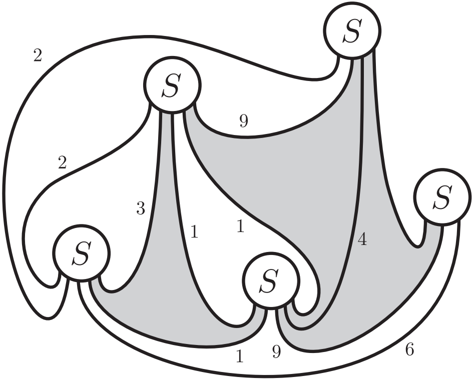
It is a (non-trivial) subfactor planar algebra.

Proof sketch: Any set of generators and relations give us a planar algebra; how do we know that $\mathcal{H}$ is a subfactor planar algebra? How do we know $\mathcal{H}$ isn't the trivial planar algebra?

Non-triviality follows from embedding $\mathcal{H}$ in a larger and easier planar algebra. We check that the image there is non-zero.

To see that $\mathcal{H}$ is a subfactor planar algebra, we need to show that $\dim(\mathcal{H}_0) = 1$. That is, how do we see that any closed diagram as a multiple of the empty diagram? Via an evaluation algorithm which treats each copy of S as a 'jellyfish' and using the substitute braiding relations to let each S 'swim' to the top of the diagram.

Begin with arbitrary closed diagram of Ss.



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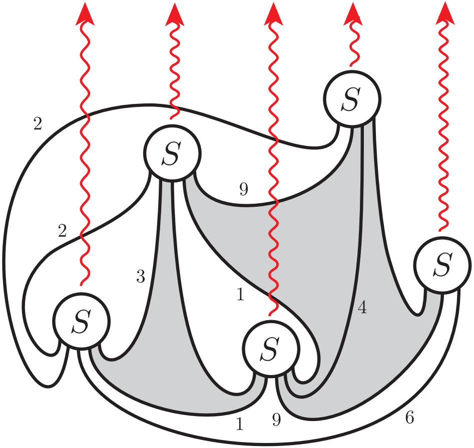

Now float each generator to the surface, using the relations.

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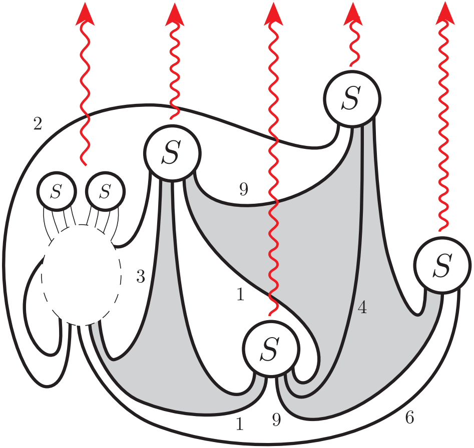
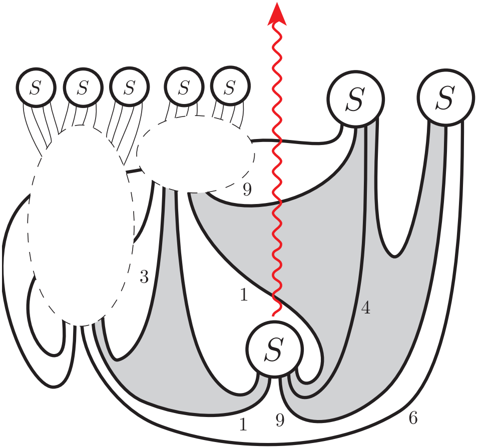


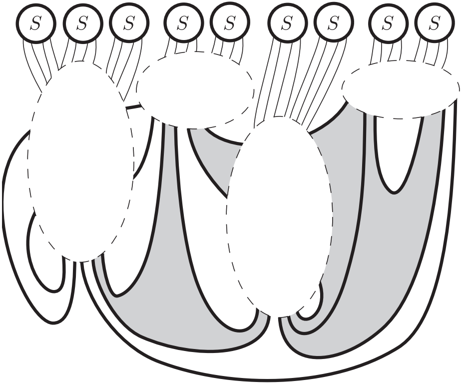
Now float each generator to the surface, using the relations.

Begin with arbitrary closed diagram of Ss.



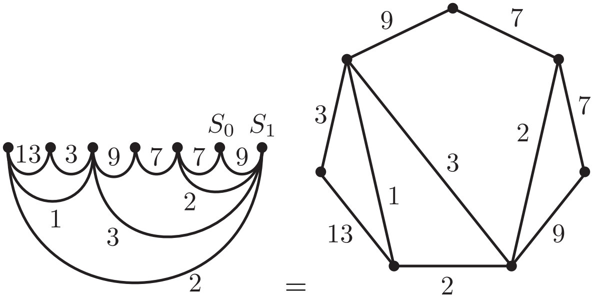
Now float each generator to the surface, using the relations.

Begin with arbitrary closed diagram of Ss.



Now float each generator to the surface, using the relations.

The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has a corner, and the generator there is connected to one of its neighbors by at least 8 edges.
- Use $S^2 \in TL$ to reduce the number of generators, and recursively evaluate the entire diagram.

The End!