NOTES FROM THE OPERATOR ALGEBRAS AND CFT WORKSHOP

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Abstract. Notes August 16-21 2010. Other people's notes appear at:
http://math.mit.edu/~eep/CFTworkshop

1. Monday, 8/16/10

1.1. **11:30am Min Ro, Hilbert spaces, polar decomposition, spectral theorem, von Neumann algebras.** Let H be a Hilbert space. $S \subset B(H)$ a set of bounded linear operators.

The commutant of S is $S' = \{a \in B(H) \mid ab = ba \, \forall b \in S\}$.

The adjoint of an operator a is the unique operator a^* that satisfies $(a\xi, \eta) = (\xi, a^*\eta)$ for all $xi, \eta \in H$. a is self-adjoint if $a^* = a$. a is normal if $a^*a = aa^*$. a is unitary if $aa^* = a^*a = 1$. a is a projection if $a^* = a = a^2$.

There is a rough dictionary, where self-adjoint operators correspond to real-valued functions, and projections correspond to characteristic functions.

Convergence: $a_\lambda \to a$ in strong operator topology if and only if we have pointwise convergence in norm: $\lim_\lambda \|a_\lambda \xi\| = \|a\xi\|$. $a_\lambda \to a$ in the weak operator topology if we have pointwise convergence in inner products: $\lim_\lambda |(a_\lambda \xi, eta)| = |(a\xi, \eta)|$ and $|(a\xi, \eta)| \leq \|a\xi\| \cdot \|\eta\|$.

Theorem (DCT) If M is a unital $*$-subalgebra of $B(H)$, then TFAE

(1) $M'' = M$
(2) M is weak operator closed.
(3) M is strong operator closed.

Definition A von Neumann algebra $M \subset B(H)$ is a unital $*$-subalgebra that satisfies $M'' = M$.

The center of M is $Z(M) = M \cap M'$.

M is a factor if $Z(M) = \mathbb{C} \cdot 1$.

Examples: $B(H)$ is a factor, and if H is finite dimensional, this is just $M_n(\mathbb{C})$. We can also take direct limits of matrix algebras, e.g., for F_n the Fibonacci sequence, take a limit of inclusions $(M_{F_{n-1}} \oplus M_{F_n} \to M_{F_n} \oplus M_{F_{n+1}}$ by $(a, b) \mapsto (b, \binom{a0}{0b}))$.

If $S \subset B(H)$ is a self-adjoint subset, then S' is a von Neumann algebra, and S'' is the smallest von Neumann algebra containing S. In particular, for any $a \in B(H)$, we can construct $W^*(a) = \{a, a^*\}''$.

Let X be a σ-finite measure space $(X = \bigcup_{i=1}^\infty E_i)$, then $L^\infty(X)$ includes into $B(L^2(X))$, and this is a von Neumann algebra. Conversely, for any commutative von Neumann algebra A, there is a measure space X such that $A \cong L^\infty(X)$.

Let X be compact, T^2. $B_b(X)$ is the space of complex Borel bounded functions. $Sp(a) = \{\lambda \in \mathbb{C} \mid \lambda \cdot 1 - a$ not invertible$\}$. This is a nonempty compact set.

Theorem: (Borel functional calculus) Let $a \in B(H)$ be normal. There is a $*$-homomorphism $B_b(sp(a)) \to W^*(a)$ taking $f \mapsto f(a)$. If $(f_n) \subset B_b(sp(a))_{sa}$, and $f_n \to f$, then $\lim f_n(a) = f(a)$.

Definition: Let X be compact Hausdorff. A spectral measure relative to (X, H) is a map E from the Borel sets of X to the projections of H such that

(1) $E(\emptyset) = 0$, $E(X) = 1$
(2) $E(\bigcup_{n=1}^\infty S_i)$ converges to $\sum_{i=1}^\infty E(S_i)$ in the strong operator topology, if S_i are pairwise disjoint.

(3) $E(S_1 \cap S_2) = E(S_1)E(S_2)$.

Definition: $\int_X f(\lambda)d(E,\lambda)$ can be defined.

Theorem: (Spectral theorem) Let $a \in B(H)$ be normal. There exists a unique spectral measure relative to $(sp(a), H)$ such that $a = \int_{sp(a)} \lambda d(E,\lambda)$.

Theorem: (Polar decomposition) Let $a \in B(H)$ with a^*a positive (i.e., its spectrum is contained in the nonnegative reals). Then there is a square root $(a^*a)^{1/2}$. There is a unique $u \in B(H)$ such that u is a partial isometry (i.e., u^*u and uu^* are projections), $a = u(a^*a)^{1/2}$, and $\ker(u) = \ker(a^*a)^{1/2}$.

If a is densely defined and closed (i.e., the graph is closed in $H \times H$), then this also works. (NB: there is also a notion of closable operator, whose graph is not closed, but has the property that the closure of the graph is the graph of something else.)

Definition: Let $S \subset B(H)$ be a self-adjoint subset. We consider two types of representation of S. (K is some other Hilbert space.)

(1) $\pi : S \to U(K)$ a group.
(2) $\pi : S \to B(K)$ a $*$-homomorphism.

We can associate a von Neumann algebra by S' or S''. We call either a factor representation if S' is a factor. This is an analogue of a piece of an isotypical decomposition - canonical, not necessarily irreducible.

If $M = S'$, then the projections in M correspond one-to-one to subrepresentations of (π, K).

Proposition: If (π, H) is a factor representation of S, and $\pi_1, K_1)$, (π_2, K_2) are two subrepresentations, then

(1) there is a unique $*$-isomorphism $\Theta : \pi_1(S)'' \to \pi_2(S)''$ such that $\Theta(\pi_1(x)) = \pi_2(x)$ for all $x \in S$.
(2) For $X = Hom_S(K_1, K_2)$, then $\overline{XK_1} = K_2$.
(3) $\Theta(a)T = Ta$ for all $a \in \pi_1(S)'', T \in X$.
(4) If $X_0 \subset X$ such that $\overline{X_0K_1} = K_2$, then $\Theta(a)$ is the unique $b \in \pi_2(S)''$ such that $bT = Ta$ for all $T \in X_0$.