SUGAWARA'S FORMULA AND THE ACTION OF $\mathrm{Diff}(S^1)$ ON POSITIVE ENERGY REPRESENTATION

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Abstract. Notes from the "Conformal Field Theory and Operator Algebras workshop," August 2010, Oregon.

G simply connected Lie Group, and $\tilde{L}G$ cecntral extension, and we're looking at positive energy representations of this. These representations are nice, and in particular one feature is that there is a projective action of $\mathrm{Diff}(S^1)$ on positive energy reps.

We will be looking at the Lie algebra $\mathrm{Lie}(\mathrm{Diff}(S^1))$ of vector fields on S^1. We can take a Fourier expansion of such things, i.e.

$$\sum_n a_n e^{in\theta} \frac{d}{d\theta}.$$

We'll restrict to polynomials. Let $L_n = -ie^{in\theta}\frac{d}{d\theta} = t^{n+1}\frac{d}{dt}$. These have commutation relations

$$[L_n, L_m] = (m-n)L_{n+m}.$$

Algebraic Geometry version: formal power series on the disk, $\mathcal{O} := \mathrm{Spec}\,\mathbb{C}(t)$.

circle $\rightsquigarrow \mathcal{K}$ formal functor disk

Consider vector fields on these $\mathrm{Der}\,\mathcal{K} = \mathbb{C}(t)\frac{d}{dt}$.

To understand the projective action of $\mathrm{Diff}(S^1)$, we have to understand the central extensions of $\mathrm{Diff}(S^1)$. There exists a universal central extension of $\mathrm{Lie}(\mathrm{Diff}(S^1))$, called the **Virasoro algebra**.

$$0 \to \mathbb{C}k \to \mathrm{Vir} \to \mathrm{Der}\,\mathcal{K} \to 0$$

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It is generated by the L_n (for $n \in \mathbb{Z}$), and a cecntral element k. The bracket now looks like

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}k.$$

A module M over Vir has central charge $c \in \mathbb{C}$ if k acts by c.

let g be a simple finite dimensionl Lie algebra. We have a cetral extension

$$0 \to \mathbb{C}I \to \widehat{g} \to g \to 0$$

and an acton of $\mathrm{Diff}(S^1)$ hich gives an action of V_{in} on $\widehat{g}$

$$[L_m, X[n]] \equiv -nX[m+n]$$

for $X \in g$ $X[n] = X \otimes t^n$.

PERs: $\widehat{LG} \rtimes \mathbb{T}_{rot}$ $V = \oplus_n V(n)$ $\frac{d}{d\theta} \in \mathrm{Lie}(\mathbb{T}_{rot})$

$$[\frac{d}{d\theta}, X[m]] = -mX[m]$$

which implies $v \in V(n)$ implies

$$X[m]v \in V(n-m)$$

Casimir: Fix an inner product on g $\langle , \rangle$. If θ is the lowest root then $\langle \theta, \theta \rangle = 2$ and if X_j is an orthonormal basis of g then

$$\Omega = \sum_j X_j^2$$

fix Ω in the center of $U(g)$ the universal enveloping algebra.

In the adjoint rep, $\Omega = 2h' \cdot I$ where h' is the dual coxeter number. In the case $G = SU(N)$ then $h' = N$. In general, if V is a highest weight representation of weight λ then

$$\Omega = \langle \lambda, \lambda + 2\rho \rangle \cdot I$$

where $\rho = \frac{1}{2}$(sum of positive roots).

Consider $\sum_{j,n \in \mathbb{Z}} X_j[n]X_j[-n]$ a natural question is if this is central and what the sum even means. Thinking like a physicist, we proceed and play with the power series to get

$$\sum_{j,n>0} (X_j[n]X_j[-n] + X_j[-n]X_j[n]) + \sum_i X_i^2$$

$$= \sum_{j,n>0} 2(X_j[n]X_j[-n]) + [X_j[-n], X_j[n]] + \sum_j X_j^2$$
$$= \sum_{j,n\in\mathbb{Z}} X_j[n]X_j[-n] = \sum_{j,n>0} [X_j[-n], X_j[n]$$

$\sum_{j,n>0}[,] = \sum_{n>0} l \cdot n \cdot dim(g) = (: X[m]Y[n] :)$ is by definition $X[m]Y[n]$ if $n \geq m$ and $y[n]X[m]$ otherwise.

Consider $\Delta_0 = \sum_{j,n\in\mathbb{Z}} : X_j[-n]X_j[n] :$. If V is a PER of LG then Δ_0 acts on V. Δ_0 is *not* in the center.

$$[Y[m], \sum_j : X_j[-n]X_j[n] :] = Z_{n+m} - Z_n$$

for $m \neq \pm n$ and is equal to $Z_{n+m} - Z_n + mY[m]I$ if $m = \pm n$ where $Z_n = \sum_{j,k} \alpha_{jk} X_k[n]X_j[m-n] :$ and $[Y, X_j] = \sum_k \alpha_{jk} X_k$ (in g).

Add all of these up. We get

$$[Y[m], \Delta_0] = m(2YI + \Omega Y)[m] = 2M(l + h')Y[m]$$
$$= 2(l + h')\frac{d}{d\theta}(Y[m]).$$

We can learn two thigs:

1.) We can get a central element $\Delta = \Delta_0 + 2(l + h')\frac{d}{d\theta}$

2.) $L_n = \frac{d}{d\theta} = -\frac{1}{2(l+h')}\Delta_0$

We can define the higher Δ_m by

$$\Delta_m = \sum_{j,n\in\mathbb{Z}} : X_j[m-n]X_j[n] :$$

Then $L_m = -\frac{1}{2(l+h')}\Delta_m$

Theorem: These satisfy the Virasaro relations with central charge

$$K = \frac{l \cdot dim(g)}{l + h'}$$