Classifying subfactors up to index 5

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joint work with Jones, Morrison, Penneys, Snyder, Tener

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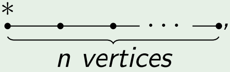
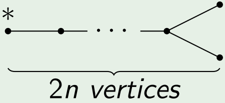
Index less than 4

Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

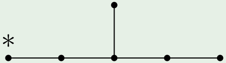
The principal graph of a subfactor of index less than 4 is one of

$A_n =$ 

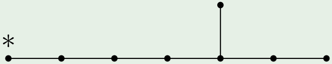
, $n \geq 2$ *index* $4\cos^2(\frac{\pi}{n+1})$

$D_{2n} =$ 

, $n \geq 2$ *index* $4\cos^2(\frac{\pi}{4n-2})$

$E_6 =$ 

index $4\cos^2(\frac{\pi}{12}) \approx 3.73$

$E_8 =$ 

index $4\cos^2(\frac{\pi}{30}) \approx 3.96$

Suppose $N \subset M$ is a subfactor, ie a unital inclusion of type II_1 factors.

Definition

The index of $N \subset M$ is $[M : N] := \dim_N L^2(M)$.

Example

If R is the hyperfinite II_1 factor, and G is a finite group which acts outerly on R, then $R \subset R \rtimes G$ is a subfactor of index $|G|$.

If $H \leq G$, then $R \rtimes H \subset R \rtimes G$ is a subfactor of index $[G : H]$.

Theorem (Jones)

The possible indices for a subfactor are

$$\{4\cos(\frac{\pi}{n})^2 | n \geq 3\} \cup [4, \infty].$$

Let $X = {}_N M_M$ and $\overline{X} = {}_M (M^{op})_N$, and $\otimes = \otimes_N$ or $\otimes_M$ as needed.

Definition

The standard invariant of $N \subset M$ is the (planar) algebra of bimodules generated by X:

$$\begin{array}{ccccccccc} X & , & X \otimes \overline{X} & , & X \otimes \overline{X} \otimes X & , & X \otimes \overline{X} \otimes X \otimes \overline{X} & , & \ldots \\ \overline{X} & , & \overline{X} \otimes X & , & \overline{X} \otimes X \otimes \overline{X} & , & \overline{X} \otimes X \otimes \overline{X} \otimes X & , & \ldots \end{array}$$

Definition

The principal graph of $N \subset M$ has vertices for (isomorphism classes of) irreducible N-N and N-M bimodules, and an edge from ${}_N Y_N$ to ${}_N Z_M$ if $Z \subset Y \otimes X$ (iff $Y \subset Z \otimes \overline{X}$).

Ditto for the dual principal graph, with M-M and M-N bimodules.

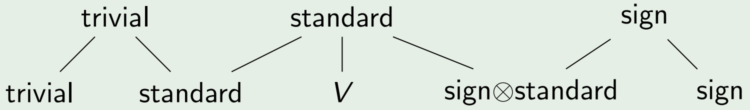
Example: $R \rtimes H \subset R \rtimes G$

Again, let G be a finite group with subgroup H, and act outerly on R. Consider $N = R \rtimes H \subset R \rtimes G = M$.

The irreducible M-M bimodules are of the form $R \otimes V$ where V is an irreducible G representation. The irreducible M-N bimodules are of the form $R \otimes W$ where W is an H irrep.

The dual principal graph of $N \subset M$ is the induction-restriction graph for irreps of H and G.

Example ($S_3 \leq S_4$)

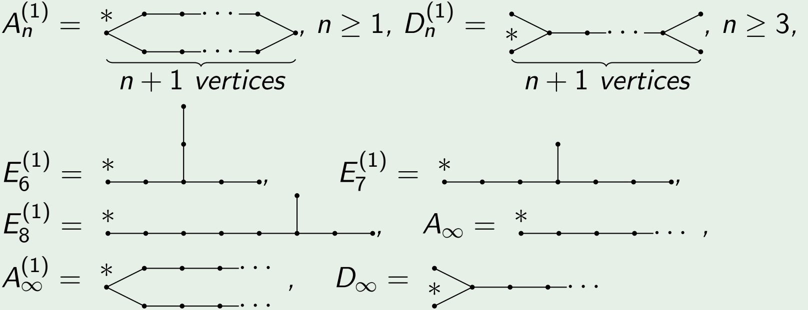


(The principal graph is an induction-restriction graph too, for H and various subgroups of H.)

Index 4

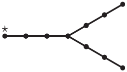
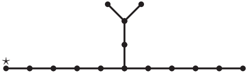
Theorem (Popa)

The principal graphs of a subfactor of index 4 are extended Dynkin diagram:



There are multiple subfactors for some of these principal graphs (eg, $n-1$ non-isomorphic hyperfinite subfactors for $D_n^{(1)}$).

Haagerup's list

- In 1993 Haagerup classified possible principal graphs for subfactors with index between 4 and $3 + \sqrt{3} \approx 4.73$:
  - , , , $\ldots$, ($\approx 4.30, 4.37, 4.38, \ldots$)
  - , (≈ 4.56)
  - , , $\ldots$ ($\approx 4.62, 4.66, \ldots$).

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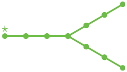
  -  ,  ,  $, \ldots,$

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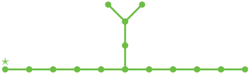
  -  , (≈ 4.56)
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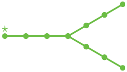
  - , , , . . . ,
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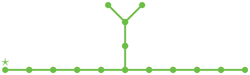
  - , (≈ 4.56)
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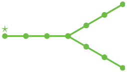
  -  ,  ,  , . . . ,
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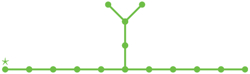
  -  , (≈ 4.56)
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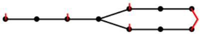
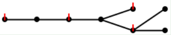
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- Last year we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

Extending the classification

We work with principal graph pairs, meaning both principal and dual principal graphs, and information on which bimodules are dual.

Example (The Haagerup subfactor's principal graph pair)

( )

The pair must satisfy an associativity test:

$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs with index below some number L up to a given rank or depth, obtaining a collection of allowed vines and weeds.

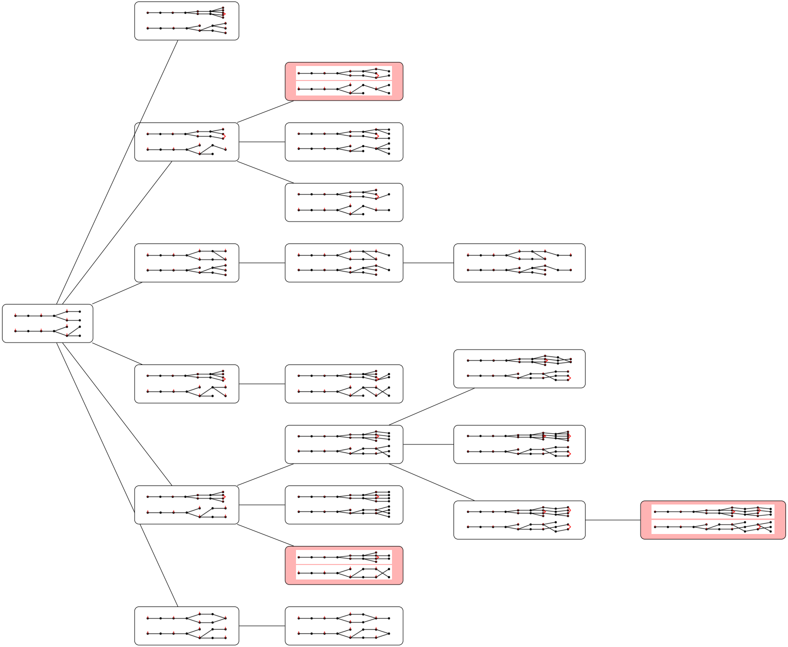
Definition

A vine represents an integer family of principal graphs, obtained by translating the vine.

Definition

A weed represents an infinite family, obtained by either translating or extending arbitrarily on the right.

We can hope that as we keep extending the depth, a weed will turn into a set of vines. If all the weeds disappear, the enumeration is complete. This happens in favorable cases (e.g. Haagerup's theorem up to index $3 + \sqrt{3}$), but generally we stop with some surviving weeds, and have to rule these out 'by hand'.



The classification up to index 5

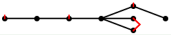
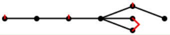
Theorem (Morrison-Snyder, part I, `arXiv:1007.1730`)

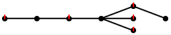
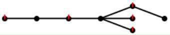
Every (finite depth) II_1 subfactor with index less than 5 sits inside one of 54 families of vines (see below), or 5 families of weeds:

$\mathcal{C} = ($  ,  $),$

$\mathcal{F} = ($  ,  $),$

$\mathcal{B} = ($  ,  $),$

$\mathcal{Q} = ($  ,  $),$

$\mathcal{Q}' = ($  ,  $).$

Theorem (Morrison-Penneys-P-Snyder, part II, `arXiv:1007.2240`)

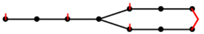
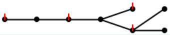
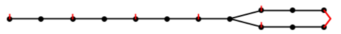
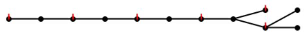
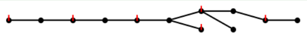
Using quadratic tangles techniques, there are no subfactors in the families $\mathcal{C}$ or $\mathcal{F}$.

Theorem (Calegari-Morrison-Snyder, `arXiv:1004.0665`)

In any family of vines, there are at most finitely many subfactors, and there is an effective bound.

Corollary (Penneys-Tener, part IV, `arXiv:1010.3797`)

There are only four possible principal graphs of subfactors coming from the 54 families

- ( )
- ( , )
- ( , )
- ( , ) .

Recent results

Theorem (Morrison-Penneys-Peters-Snyder, part V, work in progress)

There are no subfactors coming from the weed

$\mathcal{B} = ($  ,  $)$

Proof.

A connection on the principal graph only exists at a certain index (one for each supertransitivity), but the only graphs with exactly that index are certain infinite graphs which are easily ruled out. $\square$

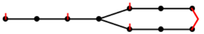
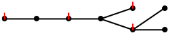
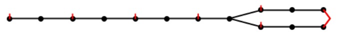
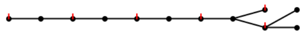
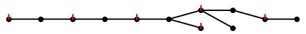
Izumi, work in progress

Also by a connection argument, the only subfactor coming from the weeds $\mathcal{Q}$ or $\mathcal{Q}'$ is 3311. ( , )

We're thus very close to completing the classification up to index 5:

Conjecture

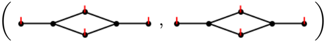
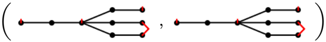
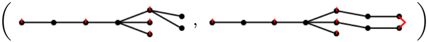
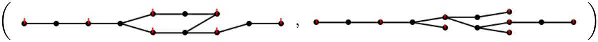
There are exactly ten subfactors other than Temperley-Lieb with index between 4 *and* 5.

- ( ),
- ( , ),
- ( , ),
- *The 3311 GHJ subfactor (MR999799), with index* $3+\sqrt{3}$ ( , ),
- *Izumi's self-dual 2221 subfactor (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$ ( , )

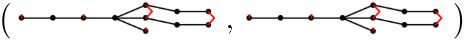
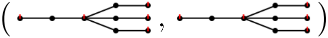
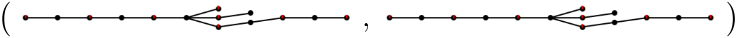
along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.

Index exactly 5

There are 5 principal graphs that come from group-subgroup subfactors, and these are known to be unique.

-  $1 \subset \mathbb{Z}/5\mathbb{Z}$
-  $\mathbb{Z}/2\mathbb{Z} \subset D_{10}$
-  $\mathbb{Z}/4\mathbb{Z} \subset \mathbb{Z}/5\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$
-  $A_4 \subset A_5$
-  $S_4 \subset S_5$

We still have a few other possibilities to rule out

- 
- 
- 

Index beyond 5

Somewhere between index 5 and index 6, things get wild:

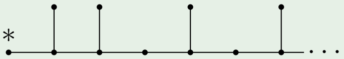
Theorem (Bisch-Nicoara-Popa)

At index 6, there is an infinite one-parameter family of irreducible, hyperfinite subfactors having isomorphic standard invariants.

and

Theorem (Bisch-Jones)

$A_2 * A_3$ is an infinite depth subfactor at index $2\tau^2 \sim 5.23607$.



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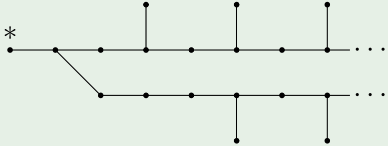
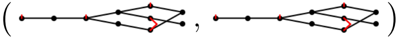
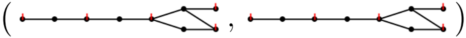
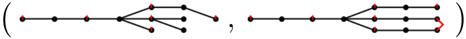
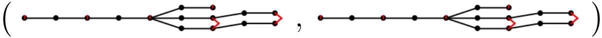
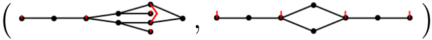
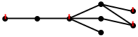
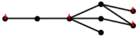
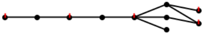
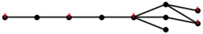


Classification above index 5 looks hard, but we can still fish for examples (only supertransitivity > 1)!
Here are some graphs that we find. (A few are previously known)

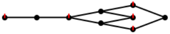
- 
  (from $SU_q(3)$ at a root of unity, index ~ 5.04892)

At index $2\tau^2 \sim 5.23607$

- 
- 
- 
- 

- ( , )
  (“Haagerup +1” at index $\frac{7+\sqrt{13}}{2} \sim 5.30278$)
- ( , ) at
  $\frac{1}{2}\left(4+\sqrt{5}+\sqrt{15+6\sqrt{5}}\right) \sim 5.78339$
- ( , ) at $3+2\sqrt{2} \sim 5.82843$

And at index 6

- ( , )
- ( , )

and several more!

The End!