Classifying subfactors up to index 5, Part I

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II_1 factors: Classification, Rigidity and Symmetry
Institut Henri Poincaré
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Suppose $N \subset M$ is a subfactor, ie a unital inclusion of type II_1 factors.

Definition

The index of $N \subset M$ is $[M : N] := \dim_N L^2(M)$.

Example

If R is the hyperfinite II_1 factor, and G is a finite group which acts outerly on R, then $R \subset R \rtimes G$ is a subfactor of index $|G|$.

If $H \leq G$, then $R \rtimes H \subset R \rtimes G$ is a subfactor of index $[G : H]$.

Theorem (Jones)

The possible indices for a subfactor are

$$\{4\cos(\frac{\pi}{n})^2 | n \geq 3\} \cup [4, \infty].$$

Let $X = {}_N L^2 M_M$ and $\overline{X} = {}_M L^2 M_N$, and $\otimes = \otimes_N$ or $\otimes_M$ as needed.

Definition

The standard invariant of $N \subset M$ is the (planar) algebra of bimodules generated by X:

$$\begin{array}{ccccccccc} X & , & X \otimes \overline{X} & , & X \otimes \overline{X} \otimes X & , & X \otimes \overline{X} \otimes X \otimes \overline{X} & , & \ldots \\ \overline{X} & , & \overline{X} \otimes X & , & \overline{X} \otimes X \otimes \overline{X} & , & \overline{X} \otimes X \otimes \overline{X} \otimes X & , & \ldots \end{array}$$

Definition

The principal graph of $N \subset M$ has vertices for (isomorphism classes of) irreducible N-N and N-M bimodules, and an edge from ${}_N Y_N$ to ${}_N Z_M$ if $Z \subset Y \otimes X$ (iff $Y \subset Z \otimes \overline{X}$).

Ditto for the dual principal graph, with M-M and M-N bimodules.

The graph norm of the principal graph of $N \subset M$ is $\sqrt{[M : N]}$.

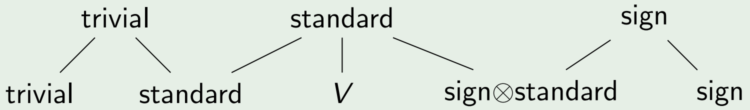
Example: $R \rtimes H \subset R \rtimes G$

Again, let G be a finite group with subgroup H, and act outerly on R. Consider $N = R \rtimes H \subset R \rtimes G = M$.

The irreducible M-M bimodules are of the form $R \otimes V$ where V is an irreducible G representation. The irreducible M-N bimodules are of the form $R \otimes W$ where W is an H irrep.

The dual principal graph of $N \subset M$ is the induction-restriction graph for irreps of H and G.

Example ($S_3 \leq S_4$)



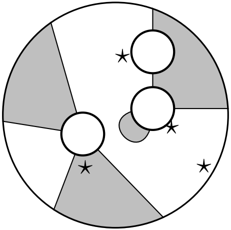
(The principal graph is an induction-restriction graph too, for H and various subgroups of H.)

Planar algebras

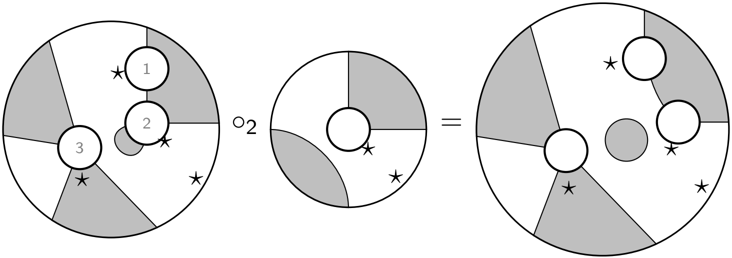
Definition

A shaded planar diagram has

- *a finite number of inner boundary circles*
- *an outer boundary circle*
- *non-intersecting strings*
- *a marked point $\star$ on each boundary circle*



We can compose planar diagrams, by insertion of one into another (if the number of strings matches up):



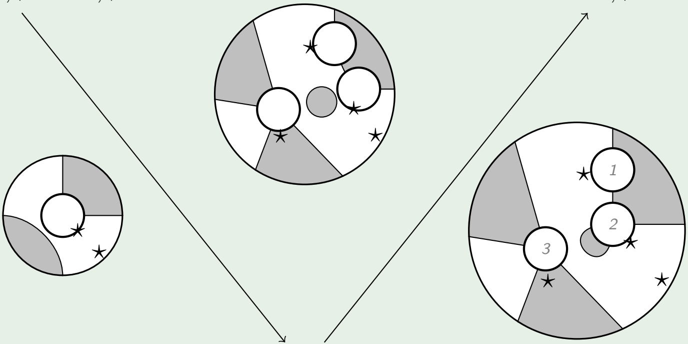
Definition

The <u>*shaded planar operad*</u> *consists of all planar diagrams (up to isomorphism) with the operation of composition.*

Definition

A planar algebra is a family of vector spaces $V_{k,\pm}$, $k = 0, 1, 2, \ldots$ *which are acted on by the shaded planar operad.*

$$
\begin{tikzcd}
V_{2,-} \times V_{1,+} \times V_{1,+} \arrow[rr] \arrow[dr] & & V_{3,+} \\
& V_{2,-} \times V_{2,+} \times V_{1,+} \arrow[ur] &
\end{tikzcd}
$$

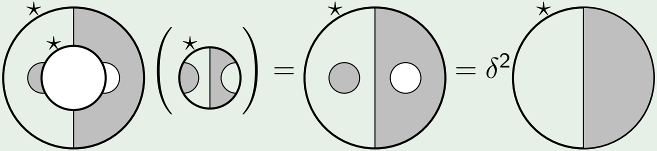


Example: Temperley-Lieb

$TL_{n,\pm}(\delta)$ is the span (over $\mathbb{C}$) of non-crossing pairings of $2n$ points arranged around a circle, with formal addition.

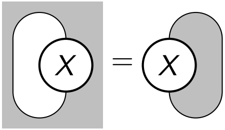
$TL_3 = \mathrm{Span}_{\mathbb{C}}\{$  $\}$.

Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings, and throwing out closed loops at a cost of $\cdot\delta$.



Subfactor planar algebras

The standard invariant of a (finite index, extremal) subfactor is a planar algebra $\mathcal{P}$ with some extra structure:

- $\mathcal{P}_{0,\pm}$ are one-dimensional
- All $\mathcal{P}_{k,\pm}$ are finite-dimensional
- Sphericality: 

- Inner product: each $\mathcal{P}_{k,\pm}$ has an adjoint $*$ such that the bilinear form $\langle x, y \rangle := \mathrm{Tr}(yx^*)$ is positive definite

From these properties, it follows that closed circles count for a multiplicative constant δ.

Definition

A planar algebra with these properties is a subfactor planar algebra.

Theorem (Jones)

The standard invariant of a subfactor is a subfactor planar algebra.

Theorem (Popa '95, Guillonet-Jones-Shlyaktenko '09)

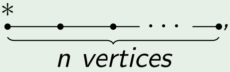
One can construct a subfactor $N \subset M$ from any subfactor planar algebra $\mathcal{P}$, in such a way that the standard invariant of $N \subset M$ is $\mathcal{P}$ again.

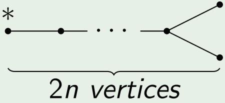
Example

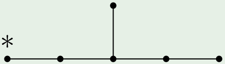
If $\delta > 2$, $TL(\delta)$ is a subfactor planar algebra. If $\delta = 2\cos(\pi/n)$, a quotient of $TL(\delta)$ is a subfactor planar algebra.

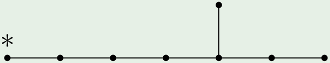
Theorem (Jones, Ocneanu, Kawahigashi, Izumi, Bion-Nadal)

The principal graph of a subfactor of index less than 4 is one of

$A_n =$  $, n \geq 2$ *index* $4\cos^2(\frac{\pi}{n+1})$

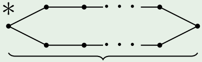
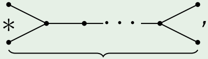
$D_{2n} =$  $, n \geq 2$ *index* $4\cos^2(\frac{\pi}{4n-2})$

$E_6 =$    *index* $4\cos^2(\frac{\pi}{12}) \approx 3.73$

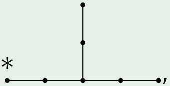
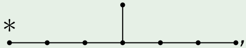
$E_8 =$    *index* $4\cos^2(\frac{\pi}{30}) \approx 3.96$

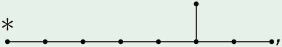
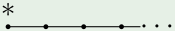
Theorem (Popa)

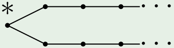
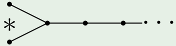
The principal graphs of a subfactor of index 4 are extended Dynkin diagram:

$A_n^{(1)} =$ , $n \geq 1$, $D_n^{(1)} =$ , $n \geq 3$,

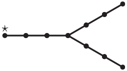
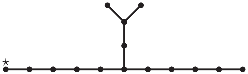
n + 1 vertices   *n + 1 vertices*

$E_6^{(1)} =$ , $E_7^{(1)} =$ ,

$E_8^{(1)} =$ , $A_\infty =$ ,

$A_\infty^{(1)} =$ , $D_\infty =$ 

There are multiple subfactors for some of these principal graphs (eg, $n-2$ non-isomorphic hyperfinite subfactors for $D_n^{(1)}$).

- In 1993 Haagerup classified possible principal graphs for subfactors with index between 4 and $3 + \sqrt{3} \approx 4.73$:
  - , , , ..., ($\approx 4.30, 4.37, 4.38, \ldots$)
  - , (≈ 4.56)
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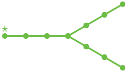
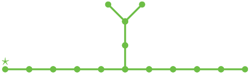
  - , , , …, ($\approx 4.30, 4.37, 4.38, \ldots$)
  - , (≈ 4.56)
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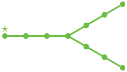
  - , , , $\ldots$, $(\approx 4.30,\ 4.37,\ 4.38, \ldots)$
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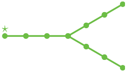
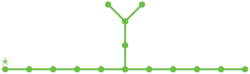
  - , , , $\ldots$, ($\approx 4.30, 4.37, 4.38, \ldots$)
  - , (≈ 4.56)
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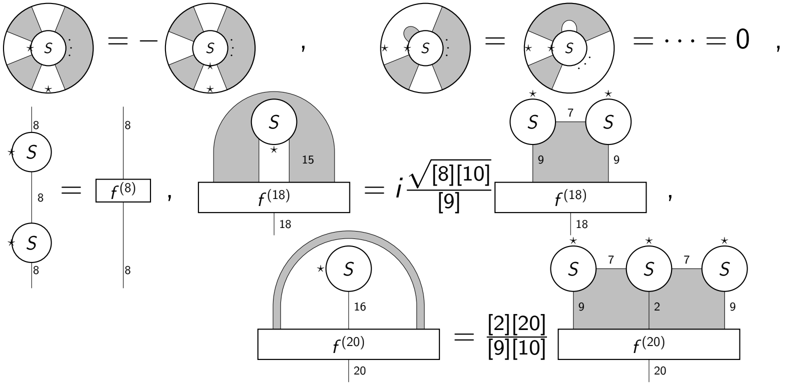
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- In 2009 we (Bigelow-Morrison-Peters-Snyder) constructed the last missing case. `arXiv:0909.4099`

The Extended Haagerup planar algebra

[Bigelow, Morrison, Peters, Snyder] The extended Haagerup planar algebra is the positive definite planar algebra generated by a single $S \in V_{8,+}$, subject to the relations  $= \delta \approx 4.377$, and



The extended Haagerup planar algebra is a subfactor planar algebra

The Extended Haagerup planar algebra redux

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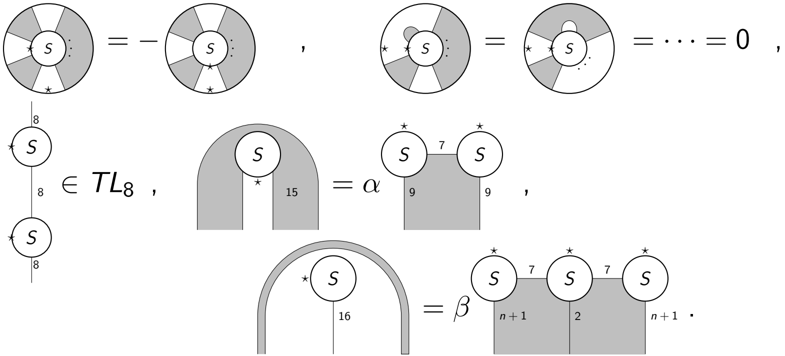


The extended Haagerup planar algebra is a subfactor planar algebra

Let V be the extended Haagerup planar algebra. How do we know $V \neq \{0\}$? How do we know $\dim(V_{0,\pm}) = 1$?

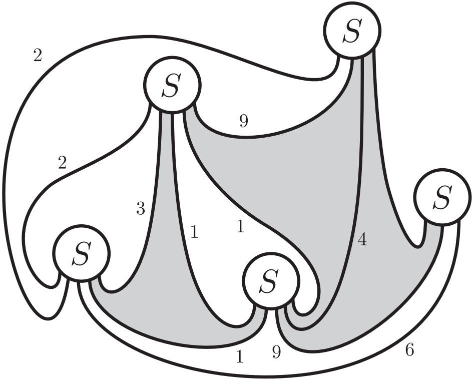
Theorem (Jones-Penneys '10, Morrison-Walker '10)

A planar algebra $\mathcal{P}$ with principal graph Γ is contained in the graph planar algebra $GPA(\Gamma)$.

We show that $V \neq \{0\}$ by finding an element S, satisfying the right relations, in the graph planar algebra of  .

Having $\dim(V_{0,\pm}) = 1$ means we can evaluate any closed diagram as a multiple of the empty diagram. We give an evaluation algorithm, which treats each copy of S as a 'jellyfish' and uses the one-strand and two-strand substitute braiding relations to let each S 'swim' to the top of the diagram.

Begin with arbitrary planar network of Ss.



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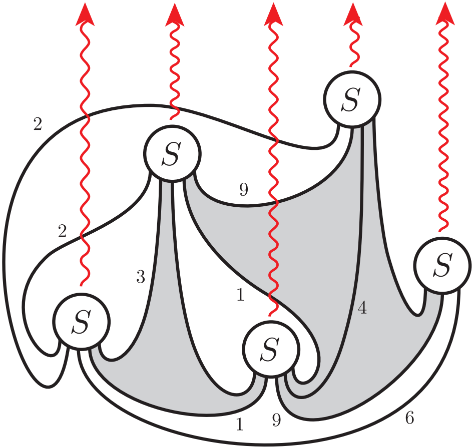

Now float each generator to the surface, using the relation.

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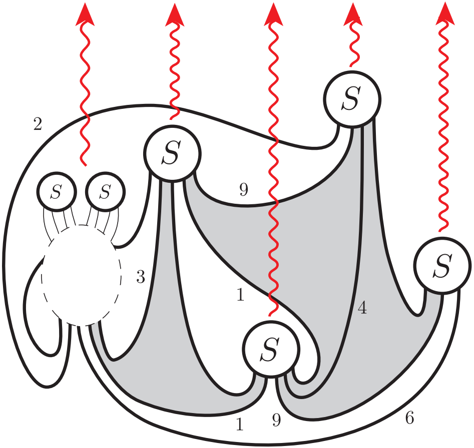

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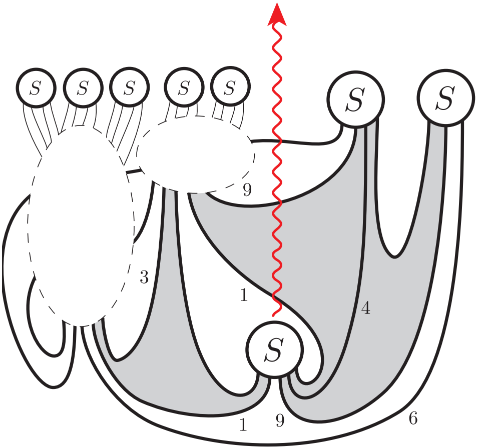

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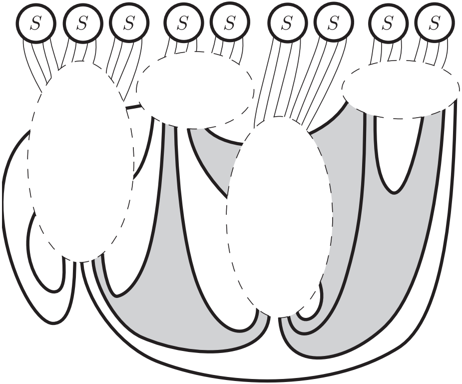
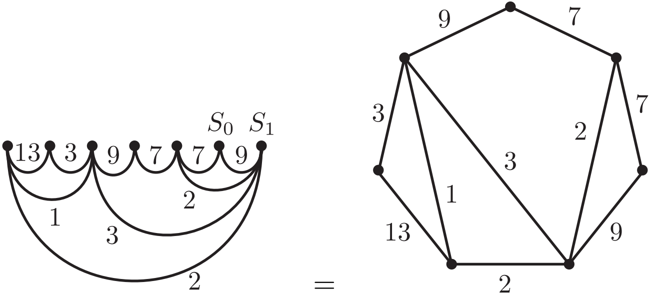


Now float each generator to the surface, using the relation.

The diagram now looks like a polygon with some diagonals, labelled by the numbers of strands connecting generators.



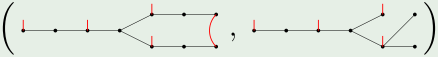
- Each such polygon has a corner, and the generator there is connected to one of its neighbours by at least 8 edges.
- Use $S^2 \in TL$ to reduce the number of generators, and recursively evaluate the entire diagram.

Extending the classification:

- Why did Haagerup stop at $3+\sqrt{3}$?
- Why try to extend it?

We work with principal graph pairs, meaning both principal and dual principal graphs, and information on which bimodules are dual to each other.

Example (The Haagerup subfactor's principal graph pair)



The pair must satisfy an associativity test:

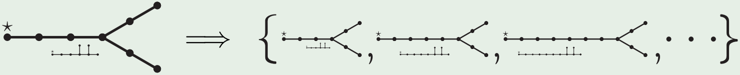
$$(X \otimes Y) \otimes X \cong X \otimes (Y \otimes X)$$

We can efficiently enumerate such pairs with index below some number L up to a given rank or depth, obtaining a collection of allowed <u>vines</u> and <u>weeds</u>.

Definition

A vine represents an infinite family of principal graphs, obtained by translating the graph.

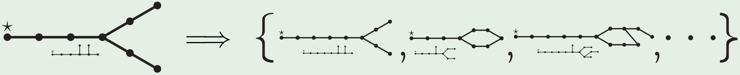
Example



Definition

A weed represents an infinite family, obtained by translating and/or extending arbitrarily on the right.

Example

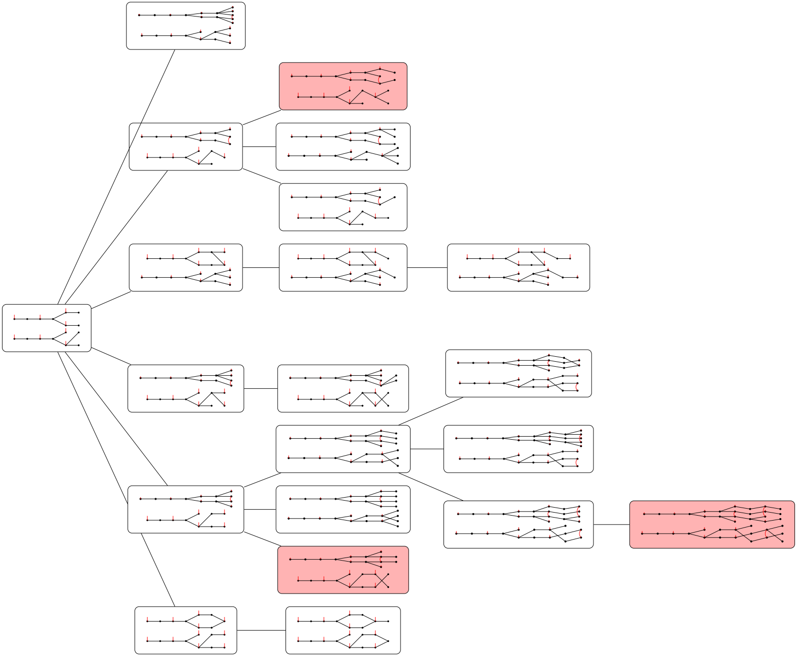


The trivial weed () represents all possible principal graphs (of irreducible subfactors).

We can always convert a weed into a vine, at the expense of finding all possible depth 1 extensions of the weed (which stay below the index limit, and satisfying the associativity condition) and adding these as new weeds.

The is a finite problem, since high valence implies large graph norm, and graph norm increases under inclusions.

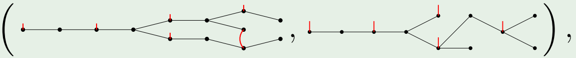
If the weeds run out, we go home happy (for example, Haagerup's classification up to $3+\sqrt{3}$). Realistically, we stop with some surviving weeds, and have to rule these out 'by hand'.

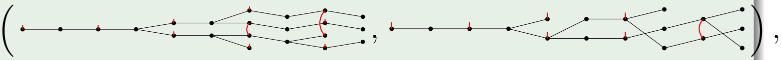


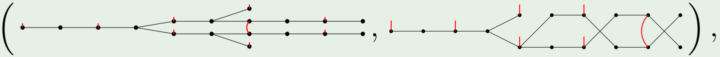
Weeds and vines for index less than 5

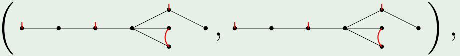
Theorem (Morrison-Snyder, part I, arXiv:1007.1730)

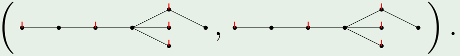
Every (finite depth) II_1 subfactor with index less than 5 sits inside one of 54 families of vines (see below), or 5 families of weeds:

$\mathcal{C} =$ 

$\mathcal{F} =$ 

$\mathcal{B} =$ 

$\mathcal{Q} =$ 

$\mathcal{Q}' =$ 

Definition

The *supertransitivity* *of a graph of an irreducible subfactor is the number of edges between its initial point and the first branch point.*

These weeds and vines are all supertransitivity three and higher. Supertransitivity one has to be dealt with separately.

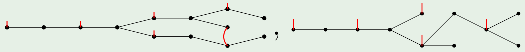
Theorem (Morrison-Snyder, part I, `arXiv:1007.1730`)

There are no subfactors with index in $(4, 5)$ *with supertransitivity one.*

Careful attention to the dimensions of the objects in the possible supertransitive-one graphs demonstrates this theorem.

Theorem (Morrison-Penneys-P-Snyder, part II, `arXiv:1007.2240`)

There are no subfactors in the family

$\mathcal{C} = \Big($  $\Big).$

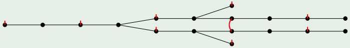
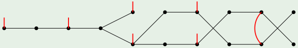
This is proved using the 'quadratic tangles' test from [Jones, '10]: For some graphs, one can deduce structure constants for a subfactor planar algebra from its principal graph:

$$r - 2 + r^{-1} = \frac{\omega + 2 + \omega^{-1}}{[m][m+2]},$$

where r is a ratio of traces, ω a root of unity, and m the depth of the branch point.

Theorem (Morrison-Penneys-P-Snyder, part II, `arXiv:1007.2240`)

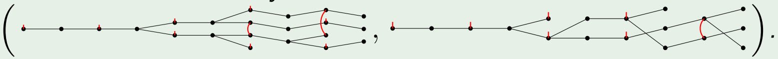
There are no subfactors in the family

$\mathcal{B} = \Big($  ,  $\Big).$

A connection on the principal graph only exists at a certain index (one for each supertransitivity). If we try to extend the graphs to reach that index, we find that the only graphs whose norm is not too small or too big have two legs which continue infinitely. This is forbidden, by [Popa '95], or [P '08] in conjunction with [Morrison-Walker '10].

Theorem (Morrison-Penneys-P-Snyder, part II, `arXiv:1007.2240`)

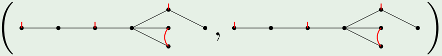
Using connections and quadratic tangles techniques, there are no subfactors in the family $\mathcal{F} =$



Theorem (Izumi-Jones-Morrison-Snyder, part III)

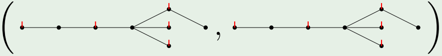
Using connections / extended quadratic tangles techniques, there are no subfactors in the families

$\mathcal{Q} =$

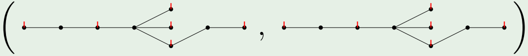


and

$\mathcal{Q}' =$



except for



Theorem (Coste-Gannon, '94)

The dimension of an object in a fusion category is a cyclotomic integer.

Corollary

The index of a finite depth subfactor is a cyclotomic integer.

Proof.

The collection of $N - N$ bimodules is a fusion category, and the dimension of M there is just the index $[N : M]$. □

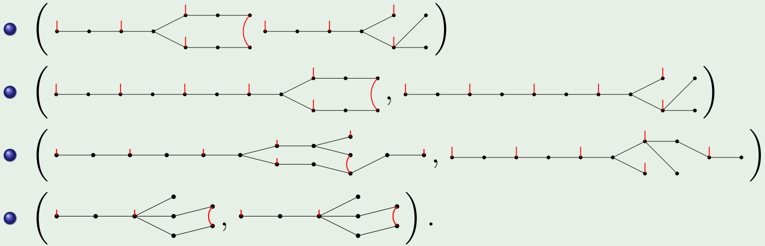
Theorem (Calegari-Morrison-Snyder, `arXiv:1004.0665`)

In any family of vines, there are at most finitely many subfactors, and there is an effective bound.

- Penneys-Tener developed algorithms for efficiently computing these bounds,
- and computed them for the 43 vines in our enumeration.
- They looked at the finitely many cases remaining from the vines, and found obstructions for all but one graph.

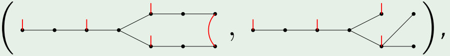
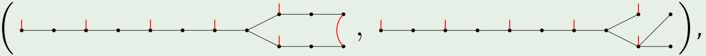
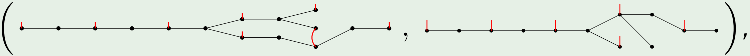
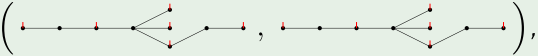
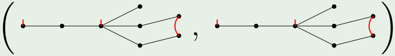
Corollary (Penneys-Tener, part IV, `arXiv:1010.3797`)

There are only four possible principal graphs of subfactors coming from the 54 vines. They are



Theorem

There are exactly ten subfactors other than Temperley-Lieb with index between 4 *and* 5.

- 
- 
- 
- *The 3311 GHJ subfactor (MR999799), with index* $3 + \sqrt{3}$
  
- *Izumi's self-dual 2221 subfactor (MR1832764), with index* $\frac{5+\sqrt{21}}{2}$
  

along with the non-isomorphic duals of the first four, and the non-isomorphic complex conjugate of the last.

Theorem (Izumi)

The only subfactors with index exactly 5 *are group-subgroup subfactors:*

- $1 \subset \mathbb{Z}_5$;
- $\mathbb{Z}_2 \subset D_{10}$;
- $\mathbb{F}_5^\times \subset \mathbb{F}_5 \rtimes \mathbb{F}_5^\times$;
- $A_4 \subset A_5$;
- $S_4 \subset S_5$.

What's next?

- Get over our dissapointment
- Try to extend the classification further
- Go fishing

Somewhere between index 5 and index 6, things get wild:

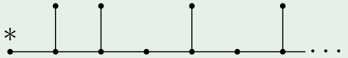
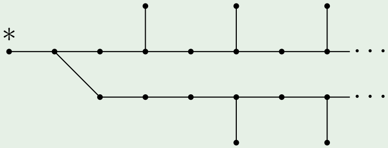
Theorem (Bisch-Nicoara-Popa)

At index 6, there is an infinite one-parameter family of irreducible, hyperfinite subfactors having isomorphic standard invariants.

and

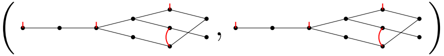
Theorem (Bisch-Jones)

$A_2 * A_3$ *is an infinite depth subfactor at index* $2\tau^2 = 3 + \sqrt{5} \sim 5.23607$.

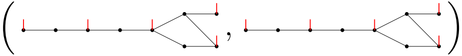
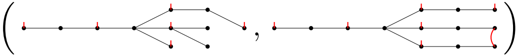
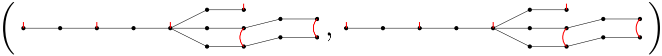
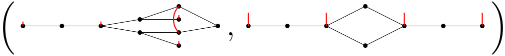

,


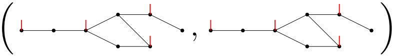
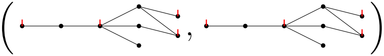
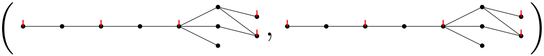
Classification above index 5 looks hard, but we can still fish for examples!

Here are some graphs that we find. (A few are previously known)

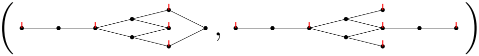
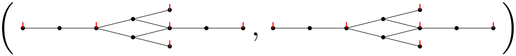
- 
  (from $SU_q(3)$ at a root of unity, index ~ 5.04892)

At index $2\tau^2 \sim 5.23607$

- 
- 
- 
- 

- 
  (“Haagerup $+1$” at index $\frac{7+\sqrt{13}}{2} \sim 5.30278$)
-  at
  $\frac{1}{2}\left(4+\sqrt{5}+\sqrt{15+6\sqrt{5}}\right) \sim 5.78339$
-  at
  $3+2\sqrt{2} \sim 5.82843$

And at index 6

- 
- 

and several more!

The End!