Constructing the Haagerup planar algebra

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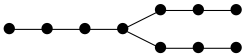
October 23, 2008

What is the Haagerup planar algebra?

The Haagerup planar algebra (HPA) is a subfactor planar algebra which has principal graph $H =$  .

Closed circles count for $\delta = \sqrt{\frac{5+\sqrt{13}}{2}} \approx 2.07$.

The Haagerup planar algebra is the smallest (in terms of δ) "exotic" planar algebra.

How to construct the Haagerup planar algebra

The Haagerup subfactor was originally constructed by Haagerup-Asaeda, and later by Izumi.

To prove the existence of the Haagerup planar algebra, without invoking the Haagerup subfactor, we

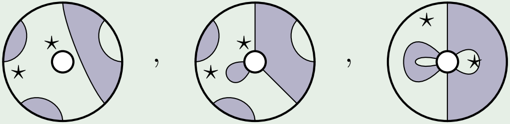
- extract information about a presentation of HPA from H.
- use this information to find a potential generator T of HPA inside $PABG(H)$.
- prove that T generates a subfactor planar algebra and this planar algebra has principal graph H.

The annular Temperley-Lieb algebra

Definition

The annular Temperley-Lieb algebra is the set of planar tangles with only one input disk.

Example



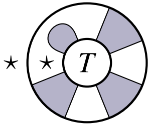
Definition

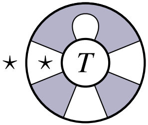
An annular Temperley-Lieb module is a family of vector spaces which has an action of the annular Temperley-Lieb algebra.

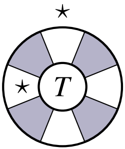
Irreducible annular Temperley-Lieb modules

An irreducible module of the annular Temperley-Lieb algebra is generated by a lowest weight element T.

If $\mathrm{wt}(T) > 0$ then we have (for some root of unity ω)

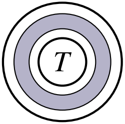


$=$


$= \cdots = 0$,


$= \omega T;$

If $\mathrm{wt}(T) = 0$ we have (for some μ, $0 \leq \mu \leq \delta$)



$= \mu^2 T.$

Combinatorics of irreducible annular Temperley-Lieb modules

Irreducible annular TL-modules are indexed by their low weight and an eigenvalue:

- $V^{0,\mu}$, for $\mu \in [0, \delta]$
- $V^{\ell,\omega}$, for $\omega^{\ell} = 1$.

Combinatorics of these modules are easy! Assuming $\delta > 2$, let's count their k-boxes:

$$\dim V_k^{0,\delta} = \frac{1}{k+1}\binom{2k}{k}, \ \dim V_k^{0,\mu} = \binom{2k}{k}, \ \dim V_k^{0,0} = \frac{1}{2}\binom{2k}{k}$$

$$\dim V_k^{\ell,\omega} = \binom{2k}{k-\ell}$$

relations from H: Annular Temperley-Lieb modules

This dimension count makes it easy to decompose a given annular Temperley-Lieb module into irreducibles. And all planar algebras are annular Temperley-Lieb modules!

Example

Suppose the Haagerup planar algebra exists. Then

k =	0	1	2	3	4	5	6
$\dim HPA_k$	1	1	2	5	15	52	199

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$\dim V_k^{0,\delta}$	1	1	2	5	14	42	132
$\dim V_k^{4,\omega_1}$	0	0	0	0	1	10	66
$\dim V_k^{6,\omega_2}$	0	0	0	0	0	0	1

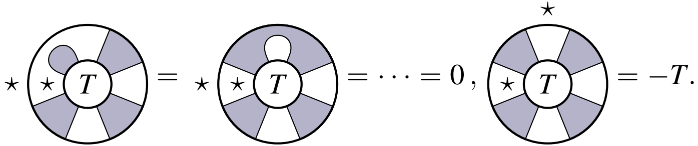
so we have

$$HPA \simeq V^{0,\delta} \oplus V^{4,\omega_1} \oplus V^{6,\omega_2} \oplus \cdots$$

Identifying a potential generator T of HPA

$$HPA \simeq V^{0,\delta} \oplus V^{4,-1} \oplus V^{6,?} \oplus \cdots$$

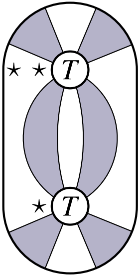
Therefore, to present the Haagerup planar algebra, we need a generator T which has eight strands, and for which



In $PABG(H)_4$ (which is 375-dimensional), we perform a similar annular Temperley-Lieb module analysis. We find there is a four-dimensional subspace of low weight four, eigenvalue -1 elements.

A potential generator T in $PABG(H)_4$

There is a unique element T in $PABG(H)_4$ satisfying these annular relations, and also



$\in \mathrm{span}\{TL, T\}$.

But how do we show that the planar algebra generated by T is the Haagerup planar algebra?

How to show $PA(T)$ is the Haagerup planar algebra

We need to show that $\dim PA(T)_0 = 1$; then we know $PA(T)$ is a subfactor planar algebra with $\delta = \sqrt{\frac{5+\sqrt{13}}{2}}$ and finite depth, and so it must have principal graph H.

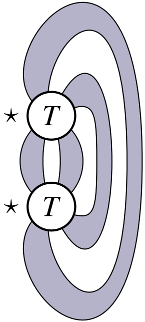
Our goal is to show that any closed diagram is a multiple of the empty diagram.

We do this by giving a short list of "quadratic" relations, and making a graph-theory argument to show these suffice.

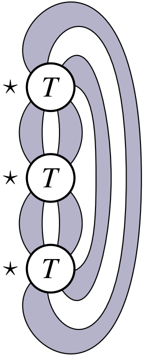
We prove these relations with linear algebra.

Traces of T^2, T^3, T^4

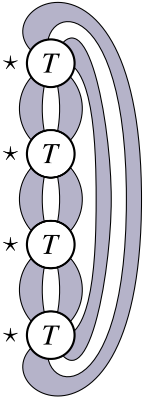
We can compute the following traces of products of Ts:



$= [5]_q,$

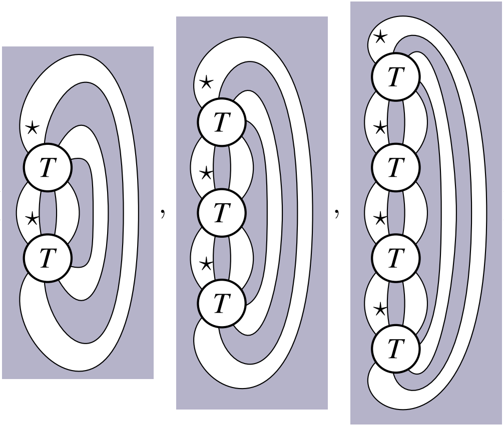


$= 0,$



$= [5]_q$

"Twisted" traces of T^2, T^3, T^4

$\{$  $\}$

$$= \{-[5]_q, i\sqrt{\frac{8(3+\sqrt{13})}{3}}, \frac{17+3\sqrt{13}}{3}\}$$

Quadratic relations

Now we use these relations, and Bessel's inequality, to show

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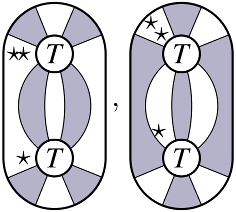

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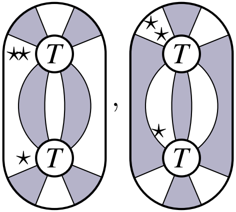
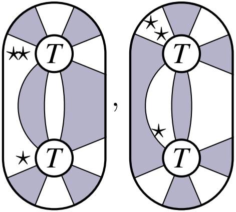


$\in \text{span}\{TL, T\}$,



$\in \text{span}\{TL, ATL_5(T)\}$,

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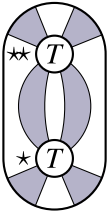

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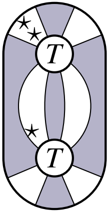

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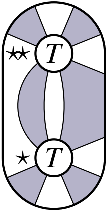

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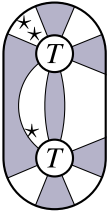

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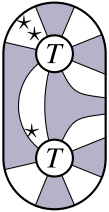

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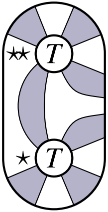

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Any closed diagram can be evaluated

Now we're ready to show

Theorem

$PA(T)_0$ is one dimensional.

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Proof by induction.

This theorem says that any closed diagram is a multiple of the empty diagram.

Any closed diagram with no Ts is a Temperley-Lieb diagram.

Any closed diagram with one T has a strand connecting T to itself.

Any closed diagram involving two or more Ts can be replaced by a diagram with fewer Ts, using the quadratic relations above. □

Reducing the number of Ts in a diagram

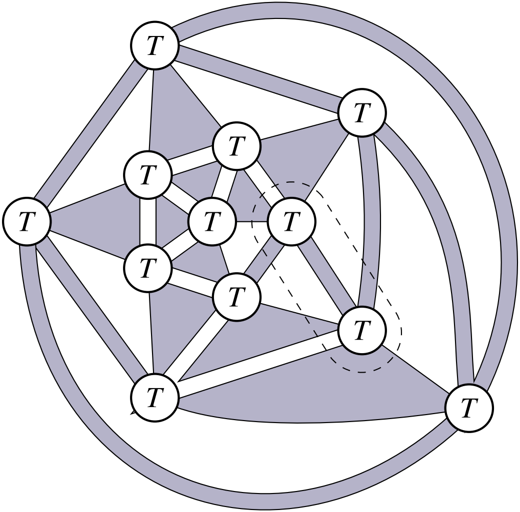
This last step – any closed diagram involving two or more Ts can be replaced by a diagram with fewer Ts, using the quadratic relations – requires some justification.

Certainly, if any two Ts in the closed diagram are attached by three adjacent strands, it is easy to see that the diagram can be replaced by [a weighted sum of] diagrams involving fewer Ts.

But what if the diagram only has Ts connected by one or two strands? Then we want to change it around until it has two Ts connected by three strands.

Creating a triple edge

A little more about this last step:



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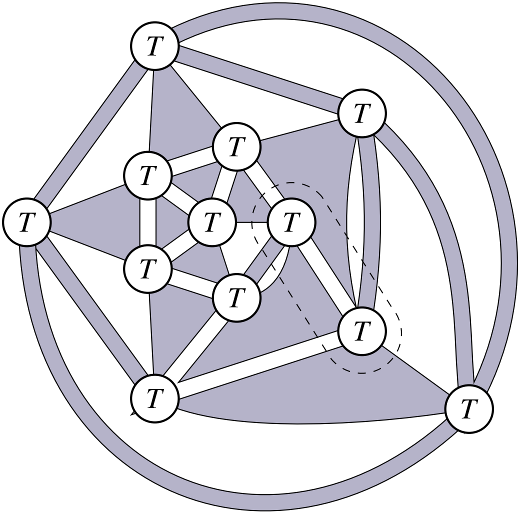
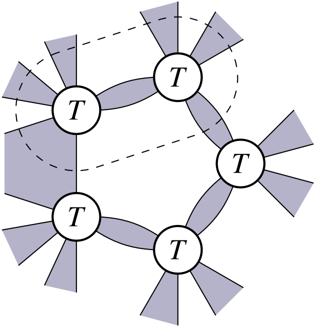


Changing a face

If we can't immediately create a triple edge, we can change the diagram until we can do so:



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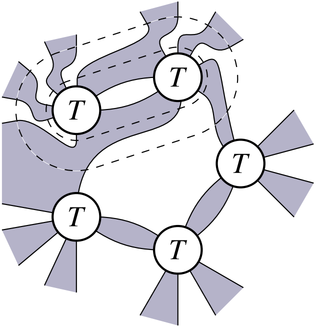

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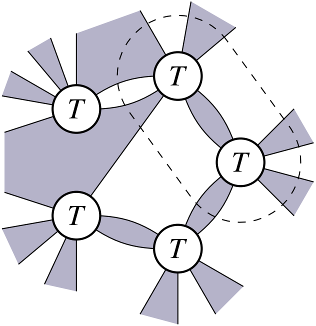

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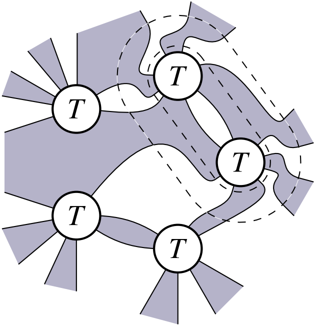

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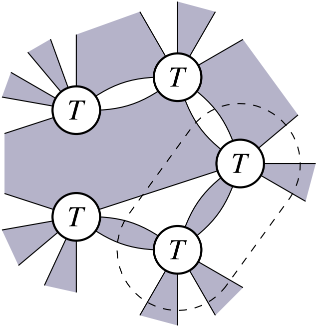

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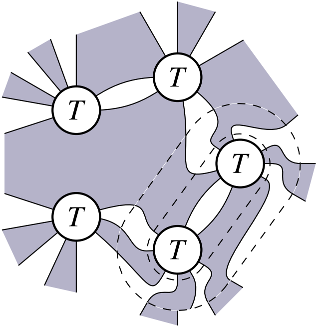

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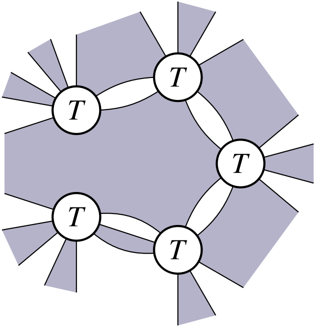
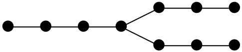


Conclusion

Therefore $PA(T)_0$ is one-dimensional, and the planar algebra generated by T is a subfactor planar algebra.

We can either directly prove it has principal graph

$H =$  or appeal to Haagerup's result on possible principal graphs, to conclude

Theorem

$PA(T)$ is a subfactor planar algebra with principal graph H.

The end

THANK YOU!