Constructing the extended Haagerup planar algebra

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joint work with Stephen Bigelow, Scott Morrison and Noah Snyder

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Outline

Invariants of subfactors

N is a subfactor of M: $\mathbf{1} \in N \subset M$ is an inclusion of II_1 factors.
Three invariants of $N \subset M$, from weakest to strongest:

- the *index* measures the relative size of N;

$$[M : N] \in \{4 \cos\left(\frac{\pi}{n}\right)^2 | n = 3, 4, 5, \ldots\} \cup [4, \infty]$$

- the *principal graph* describes tensor product rules of N,M-bimodules. If $X = {}_N M_M$, and A and B are irreducible bimodules, there is an edge from A to B if $B \subset A \otimes X$.
- the *standard invariant* is a family of algebras, with diagrammatic structure. Specifically $V_k = \mathrm{End}_{N,M}(X^{\otimes k})$.

Theorems of Popa and Jones allow us to move between subfactors and planar algebras (sometimes).

Which graphs are principal graphs?

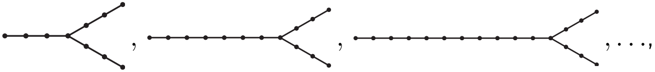
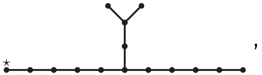
Subfactors with $[M : N] \leq 4$ have Dynkin diagrams or extended Dynkin diagrams as principal graphs.

Question

What are principal graphs for (finite-depth) subfactors with index slightly more than 4*?*

Haagerup (1994) found two families of candidates and one additional candidate, having index between 4 and $3 + \sqrt{3}$.

Classification of small-index subfactors

- Haagerup's possible principal graphs for subfactors with index less than $3+\sqrt{3}$:
  - 
  - 
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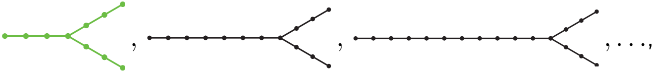
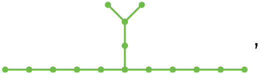
  - 
  - 
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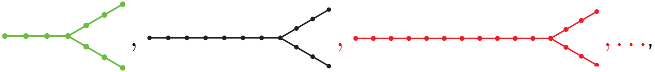
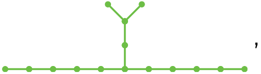
  - 
  - 
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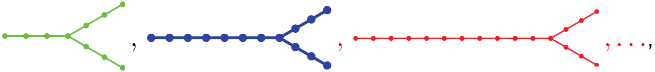
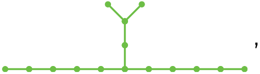
  - 
  - 
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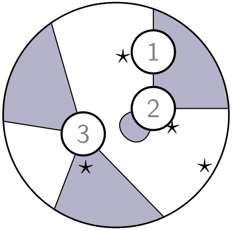
- Bisch (1998) and Asaeda & Yasuda (2007) ruled out infinite families.
- Today, we construct the missing example ('extended Haagerup'), and complete the classification.

Definition of planar algebras

A planar algebra is

- a family of vector spaces $\{V_{i,\pm}\}_{i\in\mathbb{Z}_{\geq 0}}$
- on which 'planar tangles' act; for example,



gives a map $V_{1,+} \otimes V_{2,+} \otimes V_{2,-} \longrightarrow V_{3,+}$

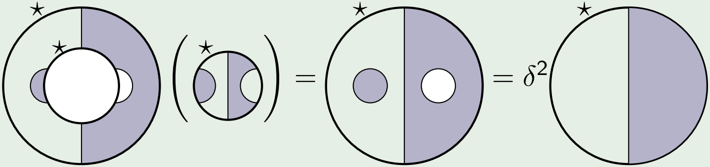
Temperley-Lieb: $TL_{n,\pm}(\delta)$ is the span (over $\mathbb{C}$) of non-crossing pairings of $2n$ points arranged around a circle, with formal addition.

Example

$TL_3 = \text{Span}_{\mathbb{C}}\{$  $\}$.

Planar tangles act on TL by inserting diagrams into empty disks, smoothing strings,and throwing out closed loops at a cost of $\cdot\delta$.

Example



The standard invariant

The *standard invariant* of a subfactor is the sequence of algebras

$$\mathrm{End}_{N,M}(X) \subset \mathrm{End}_{N,M}(X^{\otimes 2}) \subset \mathrm{End}_{N,M}(X^{\otimes 3}) \subset \cdots$$

together with its algebraic – planar algebraic – structure.

Theorem (Jones)

The standard invariant is a planar algebra.

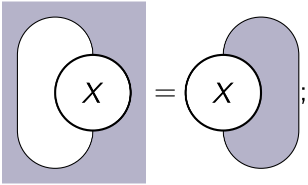
Theorem (Popa)

Subfactor planar algebras give subfactors, having the same index and principal graph.

To paraphrase, this means constructing planar algebras is equivalent to constructing subfactors.

Subfactor planar algebras

A *subfactor planar algebra* has

- $\dim V_{0,+} = \dim V_{0,-} = 1$;
- spherical trace: 

- an involution * on $V_{n,\pm}$, such that $\langle x, y \rangle = \mathrm{tr}\,(y^* x)$ is positive definite.

From these properties, it follows that closed circles count for δ, and $\delta = \sqrt{[M : N]}$.

Constructing extended Haagerup

We will give a generators-and-relations construction of the extended Haagerup planar algebra.

The first step is to find a generator inside a larger planar algebra, where calculations are straightforward.

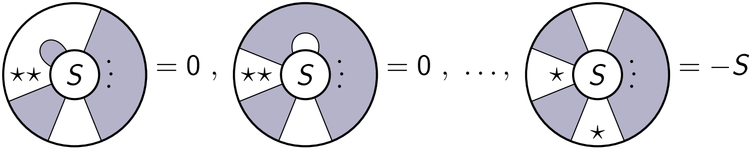
Next, we prove some relations on our generator inside this planar algebra.

Finally, we need to prove we have enough relations to guarantee that S generates a subfactor planar algebra.

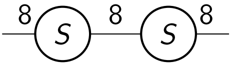
Properties of extended Haagerup

If the extended Haagerup graph is the graph of a subfactor planar algebra, what can we say about that planar algebra?

It can be singly generated as a planar algebra, by an uncappable generator S which is an eigenvector of rotation (of eigenvalue -1):



and satisfies the multiplicative relation



$= f^{(8)} \in TL_8$

Graph planar algebras

The *graph planar algebra* of a bipartite graph has

- loops based at an even/odd vertex of length $2k$ are a basis of $GPA(G)_{k,+}/GPA(G)_{k,-}$.
- the action of planar tangles is based on the concatenation of paths and the Frobenius-Perron eigenvector of G.

Though $GPA(G)$ is too big to be a subfactor planar algebra ($\dim GPA(G)_{0,+} = \#\{\text{even vertices}\} > 1$), it has a spherical trace and positive definite sesquilinear form. Further, closed circles in $GPA(G)$ count for δ, the Frobenius-Perron eigenvalue of the graph.

Theorem (Jones, unpublished)

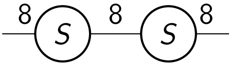
If P is a finite depth subfactor planar algebra with principal graph G, $P \hookrightarrow GPA(G)$.

Looking for an extended Haagerup generator

To construct the extended Haagerup subfactor, we start with the graph planar algebra of its principal graph eH.

$GPA(eH)_{8,+}$ is 148475-dimensional; luckily the subspace X of uncappable, $\rho = -1$ elements of $GPA(eH)_{8,+}$ is only 19-dimensional. Unluckily, it is not natural in our given basis.

We find an element $S \in X$ which further satisfies

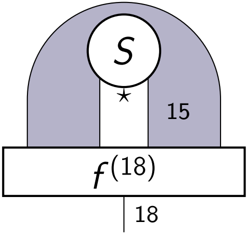


$= f^{(8)}$.

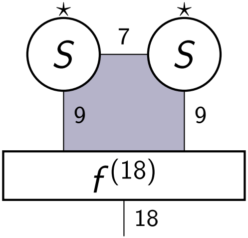
S can be written as a direct sum of 60 matrices, each of size less than 119 by 119. With computer assistance, we calculate the moments $\mathrm{tr}\left(S^2\right)$, $\mathrm{tr}\left(S^3\right)$, $\mathrm{tr}\left(S^4\right)$.

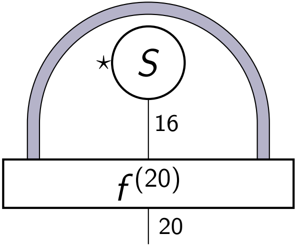
Quadratic relations

Now, we use those moments and linear algebra to prove relations:

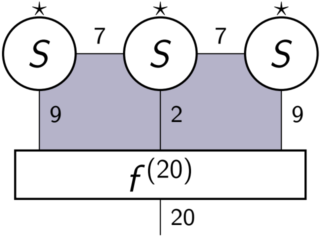


$$= i\frac{\sqrt{[8][10]}}{[9]}$$





$$= \frac{[2][20]}{[9][10]}$$



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S generates extended Haagerup

Theorem

These relations are sufficient to show that S generates the extended Haagerup planar algebra.

$GPA(eH)$ is almost a subfactor planar algebra; it is spherical and positive definite. All we are missing is $\dim V_{0,+} = \dim V_{0,-} = 1$.

If S generates a sufficiently small planar algebra (i.e., $\dim PA(S)_0 = 1$), then $PA(S)$ is a subfactor planar algebra. Further, $PA(S)$ must be the extended Haagerup planar algebra, because this is the only possible principal graph with

$$\delta = \sqrt{\frac{8}{3} + \frac{1}{3}\sqrt[3]{\frac{13}{2}\left(-5 - 3i\sqrt{3}\right)} + \frac{1}{3}\sqrt[3]{\frac{13}{2}\left(-5 + 3i\sqrt{3}\right)}} \approx 2.09218.$$

The Jellyfish algorithm, part I

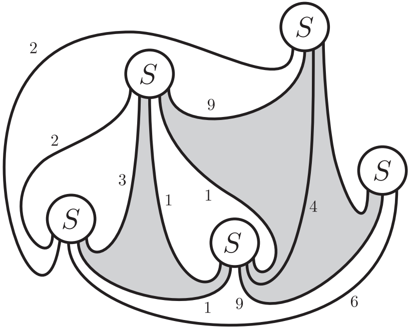
To show $PA(S)_{0,+}$ and $PA(S)_{0,-}$ are one-dimensional, we need an evaluation algorithm: a way to reduce any closed diagram in Ss to a multiple of the empty diagram.

Our algorithm works by letting the copies of our generator S 'swim to the surface,' and then removing them in pairs.

The relations we just saw let us pull S through a strand. Use this to bring all generators to the outside (multiplying if necessary).

The jellyfish algorithm in action

Begin with arbitrary planar network of Ss.



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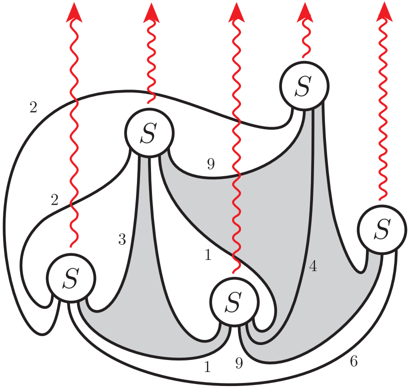

Now float each generator to the surface, using the relation.

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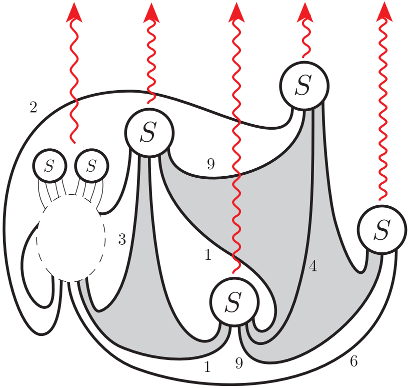

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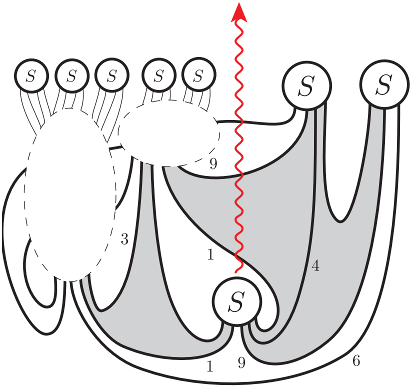

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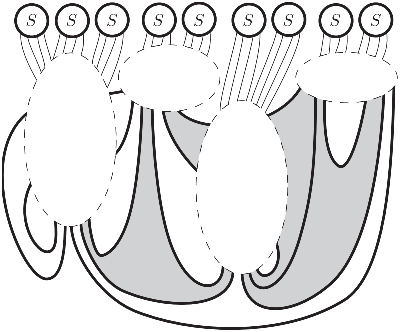

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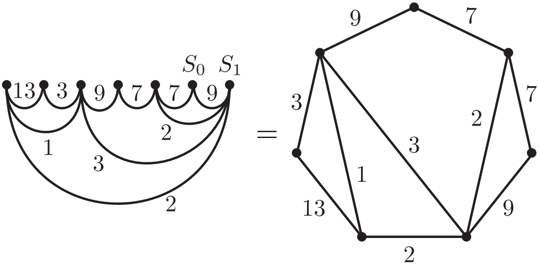
The Jellyfish algorithm, part II

Now we have a polygon with some diagonals, labelled by the numbers of strands connecting generators.



- Each such polygon has an isolated vertex. The generator there is connected to one of its neighbors by at least 8 strands.
- Use $S^2 = f^{(8)}$ to reduce the number of generators, and repeat until zero or one copies of the generator remain. The resulting picture is then in Temperley-Lieb, or zero.

The end!

In summary: S generates a subfactor planar algebra, with index $\delta^2 \approx 4.37722$; this must be the extended Haagerup planar algebra.