A Brief History of Algebra

Thomas Q. Sibley tsibley@csbsju.edu
St. John's University

We can trace the history of algebra over the entire 4000 years of recorded mathematics, even if algebra as a subject is noticeably younger. However, the advent of abstract algebra is fairly recent. Students often feel that this subject has no apparent connection to the familiar high school algebra. However, much of the historical motivation for abstract algebra is a natural continuation of the questions leading to the development of high school algebra. High school algebra topics were fully developed by 1750, after which algebra went in new and very profound directions. This essay will present some of the historical background of high school algebra and abstract algebra.

Algebra before 1750

The oldest texts we have, clay tablets from Babylonia (in present day Iraq 1950 BCE) and papyrus scrolls from Egypt (1650 BCE), already have word problems and recipes for their solutions. We would write many of these problems using first degree and second degree equations and systems of equations. Here are two examples, freely paraphrased with modern base 10 notation for ease. Note that there is no symbolic notation, no justification and no general rules. The first example uses a very early "guess and adjust" way to solve first degree equations called "the method of false position". Egyptians multiplied by doubling and divided by halving, which explains some later steps in the solution. I urge you to solve these problems using modern high school algebra before reading the provided solutions. Try to describe what characteristics you think algebra has and how many of those characteristics these examples have.

Ex. 1 (Egyptian) A quantity and one fifth of it added together become 21. What is the quantity? Provided solution: Assume the quantity were 5. 1/5 of 5 is 1. Together we get 6. How many times 6 gives 21? Twice 6 is 12, half of 6 is 3. 6 plus 12 plus 3 is 21. Then 5 plus twice 5 plus half of 5 gives 17.5, which is the desired quantity. See: 1/5 of 17.5 is 3.5 and 17.5 plus 3.5 is 21. [Katz, 28]

Ex. 2 (Babylonian) The area of a rectangle is 60 and one side exceeds the other by 7. What are the sides? Provided solution: Break in half the 7 to get 3.5. 3.5 times 3.5 is 12.25. Add the area 60 to 12.25 to get 72.25. Its square root is 8.5. Add 8.5 plus 3.5 to get 12 and take 3.5 from 8.5 to get 5. The sides are 12 and 5. [Katz, 102]

Much later, 600 BCE, the Babylonians invented a notation for zero as a place holder (but not as a separate number).

The Greeks transformed algebraic ideas into the language of geometry and even more importantly provided general proofs of results. Euclid, who lived around 300 BCE, included fairly easy algebraic properties like the following in his geometry book:

> Proposition II-4. For a line segment cut at random, the square on the whole equals the squares on the segments and twice the rectangle contained by the segments.

(In algebraic language, this becomes $(A+B)^2 = A^2 + 2AB + B^2$, where $(A+B)^2$ is "the square on the whole" and AB is "the rectangle contained by the segments" A and B.) More profoundly, Euclid generalized problems like Example 2 above. For instance, Euclid's Proposition 28 from Book VI showed how to construct a rectangle with a given side (length plus width) so that if you take off a square (whose side is the width) you get a given area. (Here is the corresponding algebra problem: For length x and width y, if $x+y = K$ and $xy = M$, find x and y.) Euclid also gave the conditions needed for a solution to be possible and a proof of this version of the quadratic formula.

Around the year 250 the Greek mathematician Diophantus used notations as algebraic shorthand, but not as symbols that can be manipulated algebraically. Around 300, the Hindus used a symbol for zero in its full numerical meaning.

The Arabic mathematician Al-Khwarizmi (825) gets credit for thinking of algebra as an actual subject: He studied and solved representative examples of types of "equations," but didn't use any symbols. In modern terms, he solved as separate types problems of the forms $ax^2 = c$, $ax^2 = bx$, $ax^2 = bx + c$, $ax^2 + bx = c$ and $ax^2 + c = bx$, where a, b, and c are all positive numbers. (Negative numbers take a while to dawn on mathematicians. Incidentally, the title of his very famous book included the Arabic word "al jabr," meaning completion, or adding the same thing to each side, from which comes our word "algebra." Al-Khwarizmi's name was corrupted into our word "algorirthm," which is what his solutions provided.) Omar Khayyam (around 1100) studied the many varieties of cubic equations, such as $ax^3 + bx^2 = cx + d$, giving geometric solutions using conics, as the Greeks had considered.

By 1545 several Italian mathematicians, culminating in Hieronimo Cardano, had amazingly found the complicated algebraic formulas to solve all cubic and fourth degree equations, still without symbols or using negative numbers as actual numbers. These formulas involved cube roots and fourth roots, as well as square roots. But sometimes square roots of negative numbers showed up in the calculations, even for real positive solutions. No one pretended then to believe these "imaginary" numbers meant anything, but geometrical justifications made them confident that their methods worked.

Raffael Bombelli in 1572 reasoned that a negative number times a negative number had to be a positive number because multiplication distributes over addition. He and others started investigating complex numbers as an algebraic system based on algebraic properties. In 1589 François Viète was the first to use symbols as things to manipulate algebraically and symbolically. The equal sign appears about this time.

In 1637 René Descartes' famous book, *Geometry*, united algebra and geometry and radically changed math. (Think of Cartesian coordinates and $y = mx + b$ and $y = ax^2 + bx + c$ as equations for lines and parabolas.) He saw the power of setting an expression equal to 0 to solve it and to investigate the number of possible positive roots. This innovation made him deal with negative roots, which he distinguished from "true roots." For example, $(x - 2)(x + 3) = 0$ has a "true root" of 2 and a "negative root" of 3.

While calculus took off full blast from Descartes' work, algebra developed more slowly. Around 1750 Leonhard Euler's textbooks set the standard for algebra notation.

It is worth considering what characterizes high school algebra. Certainly, solving equations is an important part of it. But the ability to manipulate formal symbols is vital. When we manipulate x and y, we follow formal rules. Of course, early on students think x and y are just place holders for not yet known numbers. But later students add, multiply and factor polynomials without any reference to what the abstract letters mean. All that matters are the formal properties. Formal properties lead naturally to the need for derivations and proofs. We owe the Greeks the realization that proofs are an essential part of mathematics, justifying the manipulations that others simply use.

Algebra since 1750

Finding roots of algebraic equations remained an important problem, but further progress required far more theory. Carl Friedrich Gauss for his Ph.D. in 1799 (at age 22) proved the Fundamental Theorem of Algebra: Every n^{th} degree polynomial in the complex numbers can theoretically be factored completely in the complexes and so has exactly n complex roots, counting repeats. However, his proof gives no clue how to find actual roots. Various people tried unsuccessfully to find an explicit formula to solve fifth degree equations, hoping to extend the success of the third and fourth degree formulas. Joseph Lagrange in 1770 analyzed roots of equations using permutations of the roots. These permutations (one-to-one onto functions of a set) form the prototype of a group in abstract algebra. Niels Abel in 1824 (at age 22) used this idea to prove that there can be no algebraic formula that gives the roots for all fifth degree equations. It is worth pointing out that proving something can't be done requires far more insight than finding a formula that happens to work. In 1832 Evariste Galois (just before he died at age 21) generalized this enormously as Galois theory, a major topic in a second semester abstract algebra course. Galois was able to determine which fifth and higher degree equations could be factored using roots and other algebraic operations and which couldn't be factored that way. To prove his results, Galois developed many key abstract algebra ideas—subgroups, normal subgroups, fields and more.

Mathematicians have posed and solved systems of two or three first degree equations for thousands of years. Many pieces of what we call linear algebra appear before 1750, but not in a general setting or fitting together: the Chinese (200) invented matrices to solve systems, separately Colin Maclaurin (1729) invented determinants to solve systems, and Euler (1748) starts developing the idea of eigenvalues for linear systems. After 1830 these threads get woven together into the general subject we call linear algebra, especially by Arthur Cayley. Cayley considered matrices as functions. He also used matrices and vectors as algebraic objects, working with equations whose variables and constants can be vectors or matrices. He stated and made use of formal algebraic properties of these objects. He gave the abstract definition of a group (1849), independent of the objects in it, starting the emphasis in math on abstract systems.

In 1872 Felix Klein convinced geometers of the power of transformations (permutations), such as rotations and reflections. A geometry or a geometrical structure has its own set of transformations, which always forms a group. Thus groups quickly became essential in investigating geometry. Others used this approach in analysis and topology. In 1891 Vyatseglav Fedorov, a Russian chemist and mathematician, used groups to classify all possible chemical crystals before x-rays could reveal how atoms were arranged. In the 1920s physicists realized that groups (and linear algebra) were essential to study quantum mechanics. Groups have such widespread theoretical and practical importance that mathematicians realize that all mathematics majors need to understand a fair amount about them. Many other algebra topics appear in other areas of mathematics, as well as in physics, computer science and other disciplines.

After 1850 algebraists started focusing on abstract systems satisfying formal properties, although initially as a way to systematize already known examples. Bit by bit they realized that formal proofs about abstract systems are often clearer, simpler and more general than ones about specific systems, even when the system is familiar. Emmy Noether (1921) deserves credit for reformulating algebra as the study of abstract structure, which will be our approach. Arbitrary systems, their properties, interrelations and classification are now major concerns of algebra. This abstract approach has led to new and deeper insights about algebra and all of mathematics. Algebra is the first place in the undergraduate curriculum that can showcase the power of the formal abstract approach. Understanding the power of abstraction and general proofs is an additional reason for requiring all mathematics students to study this beautiful subject.

Bibliography

Katz (ed.), *The Mathematics of Egypt, Mesopotamia, China, India and Islam: A Sourcebook*, Princeton: Princeton University Press, 2007.

Kline, *Mathematical Thought from Ancient to Modern Times*, New York: Oxford University Press, 1972.

Struik, *A Concise History of Mathematics*, New York: Dover, 1967.