Discrete Functional Analysis

Martin Buntinas
Loyola University Chicago

Functional Analysis is an abstract branch of mathematics based on the theory of topology. However, some of the most important applications have a discrete setting, such as the theory of Fourier series, sequence space, summability, approximation theory, and matrix transformations. In such a discrete setting, the subject is accessible to students who have merely a background of real analysis and linear algebra.

The topics of this conference are listed as sequence spaces, series, summability theory, Hausdorff transformations, related topics, and applications. It is sometimes difficult to publish and receive funding in these topics because they are considered by many to be closed, even though the area has recently experiencing increased research activity. The subject of Discrete Functional Analysis encompasses all of these topics.

What is in a name is important because it helps us see our work in a larger context and it describes our work to the outside more clearly. For this reason, I propose that we think of the area of our research as Discrete Functional Analysis.

This talk gives an overview of the most important topics and results in the area of Discrete Functional Analysis, some history, as well as some recent results.

Discrete Functional Analysis

Martin Buntinas
Loyola University Chicago

Functional Analysis is an abstract branch of mathematics based on the theory of topology. Some of the more esoteric topics include

- Nets, filters, ultafilters
- Moore-Smith convergence
- Projective limit topolgies, inductive limit topologies
- Bornological spaces
- Uniform spaces
- Vector valued integrals
- Nuclear mapping and nuclear spaces

However, some of the most important applications have a discrete setting. These topics include

- Theory of Fourier series
- Sequence space
- Summability theory
- Approximation theory
- Matrix transformations

The topics of this conference are listed as

- Sequence spaces
- Series
- Summability theory
- Hausdorff transformations
- Related topics and applications

It is sometimes difficult to publish and receive funding in these topics because they are considered by many to be closed, even though the area has recently experiencing increased research activity. There are 40 talks at the present workshop.

What is in a name is important because it helps us see our work in a larger context and it describes our work to the outside more clearly. For this reason, I propose that we think of the area of our research as

Discrete Functional Analysis

This encompasses almost all of talks of this conference.

An added advantage is that Discrete Functional Analysis is accessible to students who have merely a background of real analysis and linear algebra.

Here I give an overview of some important topics in the area of Discrete Functional Analysis, some history, as well as demonstrate how it accessible to students with only a background in real analysis and linear algebra.

Every student of Calculus learns about

Taylor Series Representation of Functions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\widehat{e^x} = (1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots, \frac{1}{n!}, \ldots)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\widehat{\sin x} = (0, 1, 0, \frac{-1}{3!}, 0, \frac{1}{5!}, 0, \frac{-1}{7!}, \ldots)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\widehat{\cos x} = (+1, 0, \frac{-1}{2!}, 0, \frac{+1}{4!}, 0, \frac{-1}{6!}, \ldots)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots, \text{ for } |x| < 1$$

$$\widehat{\frac{1}{1+x}} = (1, -1, +1, -1, +1, -1, \ldots), \text{ for } |x| < 1$$

Taylor representations of functions as sequences is very important theoretically. However, they work only for functions $f(x)$ that are infinitely differentiable around 0 since for

$$\widehat{f(x)} = (a_0, a_1, a_2, a_3, \ldots) \text{ we have } a_n = \frac{f^{(n)}(0)}{n!}$$

For applications to Science, Technology, and Engineering, another form of representation is important.

Fourier Series Representation of Functions

There are two forms:

$$\text{Complex form: } f(x) \sim \sum_{n=-\infty}^{+\infty} c_n e^{inx}$$

$$\text{Real form: } f(x) \sim \sum_{n=0}^{+\infty} a_n \cos nx + \sum_{n=1}^{+\infty} b_n \sin nx$$

For example, for even functions,

$$f(x) = f(-x) \text{ defined on interval } [-\pi, +\pi],$$

we have, in the real form, $b_n = 0 \ \forall n = 1, 2, 3, \ldots$

$$f(x) \sim \sum_{n=0}^{+\infty} a_n \cos nx$$

For the sake of simplicity here, we consider only even functions. The theory of Fourier series shows that

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \ dx$$

Some Examples of Fourier series for $-\pi \leq x < \pi$

$$x^2 = \frac{-\pi^2}{3} - \frac{4}{1^2}\cos x + \frac{4}{2^2}\cos 2x - \frac{4}{3^2}\cos 3x + \cdots$$

$$\widehat{x^2} = (\frac{-\pi^2}{3}, \frac{-4}{1^2}, \frac{+4}{2^2}, \frac{-4}{3^2}, \frac{+4}{4^2}, \ldots)$$

$$|x| = -1 + \frac{4}{1^2}\cos 2x + \frac{4}{4^2}\cos 4x + \frac{4}{6^2}\cos 6x + \cdots$$

$$\widehat{|x|} = (-1, 0, \frac{4}{2^2}, 0, \frac{4}{4^2}, 0, \frac{4}{6^2}, 0, \frac{4}{8^2}, \ldots)$$

$$f(x) = \begin{cases} -\frac{1}{2} & \text{if} \quad -\pi \leq x < 0 \\ +\frac{1}{2} & \text{if} \quad 0 < x < \pi \end{cases}$$

$$\widehat{f} = (0, \frac{-2}{\pi}, 0, \frac{+2}{3\pi}, 0, \frac{-2}{5\pi}, 0, \frac{+2}{7\pi}, \ldots)$$

Both Taylor theory and Fourier theory show us that function spaces can be represented as sequence spaces of Taylor and Fourier coefficients. This leads to one of the topics of Discrete Functional Analysis:

Theory of Sequence Spaces

S. Banach in 1932,
G. Köthe and O. Toeplitz in 1934,
K. Zeller in 1950,
D.J.H. Garling in 1967.

The space of all real or complex valued sequences:

$$\omega = \{x = (x_0, x_1, x_2, \ldots, x_n, \ldots) \mid x_n \in \mathbb{R} \text{ or } \mathbb{C} \ \forall n \ \}$$

How do we do analysis on ω?

In Real Analysis the points $x \in \mathbb{R}$ have an absolute value $|x|$. Also the distance between points $x, y \in \mathbb{R}$, is $d(x, y) = |x - y|$.

In ω, the 'points' are sequences $x = (x_0, x_1, x_2, \ldots, x_n, \ldots)$. We can define an analogous norm $\|x\|$ and a distance between $x, y \in \omega$ as $d(x, y) = \|x - y\|$

Some Examples of norms and distance functions:

$$\|x\|_\infty = \sup_n |x_n|, \quad d(x,y) = \|x-y\|_\infty = \sup_n |x_n - y_n|$$

$$\|x\|_1 = \sum_{n=0}^{\infty} |x_n|, \quad d(x,y) = \|x-y\|_1 = \sum_{n=0}^{\infty} |x_n - y_n|$$

$$\|x\|_2 = \sqrt{\sum_{n=0}^{\infty} |x_n|^2}, \; d(x,y) = \|x-y\|_2 = \sqrt{\sum_{n=0}^{\infty} |x_n - y_n|^2}$$

Note: These norms do not apply to all of ω. We need to restrict to subspaces E of ω called Sequence Spaces.

Conditions on Sequence Spaces E

1. E is a linear subspace of ω.
2. E has a norm satisfying
   (a) $\|x\| = 0 \iff x = 0 = (0,0,0,\ldots) \in E$
   (b) $\|cx\| = |c| \cdot \|x\|, \; \forall c \in \mathbb{R}$ or $\mathbb{C}, x \in E$
   (c) $\|x+y\| \le \|x\| + \|y\|, \; \forall x, y \in E$
3. E is complete with respect to the norm. That is, Cauchy sequences of elements in E converge in E:
   $\|x^n - y^m\| \to 0 \; (n, m \to \infty) \Rightarrow$
   $\exists \; x \in E \ni \|x - x^n\| \to 0 \; (n \to \infty)$
4. Coordinate functionals $f_n : x \to x_n$ are continuous.
   That is, $\forall \; n, \|x\| \approx 0 \Rightarrow x_n \approx 0$.
   Equiv., $\forall n = 0, 1, 2, \ldots, \exists M_n \ni |x_n| \le M_n \|x\|, \forall x \in E$

We say that E is a BK-space if it satisfies the above conditions.

More generally, sequence spaces can be FK-spaces. FK-spaces are complete metrizable sequence spaces with continuous coordinate functionals. They are more general than BK-spaces but have almost all of the properties of BK-spaces. In this talk, we consider only BK-spaces.

Some examples of BK-Spaces

$$\ell^1 = \{x \in \omega \mid \sum_{n=0}^{\infty} |x_n| < \infty\}, \quad \|x\|_1 = \sum_{n=0}^{\infty} |x_n|$$

$$\ell^2 = \{x \in \omega \mid \sum_{n=0}^{\infty} |x_n|^2 < \infty\}, \quad \|x\|_2 = \sqrt{\sum_{n=0}^{\infty} |x_n|^2}$$

$$c_0 = \{x \in \omega \mid \lim_{n\to\infty} x_n = 0\}, \quad \|x\|_\infty = \sup_n |x_n|$$

$$\ell^\infty = \{x \in \omega \mid \sup_n |x_n| < \infty\}, \quad \|x\|_\infty = \sup_n |x_n|$$

$$cs = \{x \in \omega \mid \sum_{n=0}^{\infty} x_n \text{ exists}\}, \quad \|x\|_{cs} = \sup_N \Big|\sum_{n=0}^{N} x_n\Big|$$

$$bs = \{x \in \omega \mid \|x\|_{cs} = \sup_N \Big|\sum_{n=0}^{N} x_n\Big| < \infty\},$$

$$bv = \{x \in \omega \mid \|x\|_{bv} = \sum_{n=0}^{N} |x_n - x_{n+1}| + \sup_n |x_n| < \infty\}$$

$$bv_0 = bv \cap c_0$$

Spaces of Fourier coefficients can form BK-spaces.

The study of function spaces as sequence spaces of Fourier coefficients was pioneered by Günther Goes.

For example, for continuous functions,

$$\widehat{C} = \{\widehat{f} = (a_0, a_1, a_2, \ldots) \mid f \text{ is continuous on } [-\pi, \pi]\}$$

Here $a_n = \widehat{f}(n) = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x) \cos nx \; dx$ and the norm is $\|\widehat{f}\|_C = \displaystyle\sup_{-\pi \le x \le \pi} |f(x)|$.

$$\widehat{L} = \{\widehat{f} \mid f \text{ is Lebesgue integrable on } [-\pi, \pi]\}$$

with norm $\|\widehat{f}\|_L = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f(x)| dx.$

$$\widehat{L^2} = \{\widehat{f} \mid f^2 \text{ is Lebesgue integrable on } [-\pi, \pi]\}$$

with norm $\|\widehat{f}\|_2 = \sqrt{\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} |f^2(x)| dx.}$

$$\widehat{BV} = \{\widehat{f} \mid f(x) = g(x) - h(x), g \uparrow, h \uparrow \text{ on } [-\pi, \pi]\}$$

with norm

$$\|\widehat{f}\|_{BV} = \sup \sum_{i=0}^{n} |f(x_i) - f(x_{i-1})|$$

over all partitions *P*: $-\pi = x_0 < x_1 < x_2 < \cdots < x_n = \pi.$

An important property of sequence spaces recognized by K. Zeller is Sectional Convergence AK.

If E is a BK-space and $x \in E$, we define the n-th section of $x = (x_0, x_1, x_2, \ldots)$ as the finite sequence $s^n x = (x_0, x_1, x_2, \ldots, x_n, 0, 0, 0, \ldots)$.

A BK-space has the property AK if, for all $x \in E$, we have $s^n x \to x$ as $n \to \infty$.

That is, $\|x - s^n x\| \to 0$ as $n \to \infty$.

This property was introduced by K. Zeller in 1951. There have been many papers written on this including a very important one by D.J.H. Garling in 1967.

A problem for Fourier series

BK-spaces of Fourier coefficients generally do not have the property AK.
(There is an important exception $\widehat{L^2} = \ell^2$)

This is because the Fourier series of a function does not have to converge to the function. This was a BIG problem in 19th century. However, in 1904, L. Fejér showed that the Cesàro means of Fourier series converge to the function.

Günther Goes recognized that one has to look at Cesàro summability in BK-spaces instead of the property AK.

If we define the n^{th} Cesàro section of a sequence x as

$$\sigma^n x = \frac{s^0 x + s^1 x + \cdots + s^n x}{n+1},$$

where

$$s^k x = (x_0, x_2, ..., x_k, 0, 0, 0, \ldots),$$

then we say that a BK-space E has the property σK if for all $x \in E$, we have $\sigma^n x \to x$ (as $n \to \infty$). That is, $\|x - \sigma^n x\| \to 0$ as $n \to \infty$. If a space has the property AK, then it has σK, but not the other way around. For example, the BK-spaces $\widehat{L}$ and $\widehat{C}$ have σK but not AK.

This leads to another of the major topics of Discrete Functional Analysis:

Summability Theory

Summability theory is a rich field. Many of the talks given at this conference deal with various methods of summability, such as Hausdorff summability.

I will now consider three ways of combining BK-spaces and their relationship to summability in BK-spaces.

1. Sum of BK-spaces.

If E and F are BK spaces, we define

$$E + F = \{x + y \in \omega \mid x \in E \text{ and } y \in F\}$$

This is of interest in itself, but especially important in the study of the dual space of the intersection of BK-spaces because for the dual space we have

$$(E \cap F)' = E' + F'$$

Here is an example of a summability result that uses the sum of BK-spaces, discovered by Günther Goes:

If

$$|\sigma c| = \{x \in \omega \mid \sum_{n=0}^{\infty} |\frac{s_n}{n} - \frac{s_{n+1}}{n+1}|\}$$

is the space of all absolutely Cesàro limitable sequences and

$$dl = \{x \in \omega \mid \sum_{n=0}^{\infty} |\frac{x_n}{n}| < \infty\},$$

then

$$|\sigma c| = bv + dl$$

2. Product of BK-spaces.

In Fourier analysis, the convolution of functions is an important operation.

$$h(x) = f \star g(x) = \int_{-\pi}^{\pi} f(t)g(x-t)\ dt$$

It turns out that for convolutions, the sequence of Fourier coefficients multiply:

For $\widehat{f} = (a_0, a_1, a_2, \ldots)$ and $\widehat{g} = (b_0, b_1, b_2, \ldots)$ we have

$$\widehat{f \star g} = \widehat{f} \cdot \widehat{g} = (a_0 b_0, a_1 b_1, a_2 b_2, \ldots)$$

This makes it very important to study products of sequences of Fourier coefficients.

Günther Goes and I wrote two introductory papers on products of BK-spaces.

If E and F are BK-spaces, the coordinatewise product of E and F is

$$E \cdot F = \{x \cdot y = (x_0 y_0, x_1 y_1, x_2 y_2, \ldots) \mid x \in E, y \in F\}$$

Unfortunately, $E \cdot F$ need <u>not</u> be a linear space. And even if it is a linear space, it need not be a BK-space.

For example, $\ell^1 \cdot \ell^1 \subsetneq \ell^1$

Define the BK-product of E and F as

$$E \widehat{\otimes} F = \bigcap \{W \mid W \text{ is a BK-space and } E \cdot F \subset W\}$$

For example,

$$\ell^1 \widehat{\otimes} \ell^1 = \ell^1$$

$$cs \widehat{\otimes} cs = c_0$$

$$bs \widehat{\otimes} bs = \ell^\infty$$

3. Multipliers.

If E and F be BK-spaces, the multiplier space from E to F is defined as follows.

$$(E \to F) = \{x \in \omega \mid x \cdot y \in F \ \forall y \in E\}$$

For example, Köthe and Toeplitz studied α duals in 1934 defined as

$$E^\alpha = (E \to \ell^1)$$

$$\text{For example, } (\ell^1)^\alpha = (\ell^1 \to \ell^1) = \ell^\infty$$

$$\text{and } c_0^\alpha = (c_0 \to \ell^1) = \ell^\infty$$

Later the β dual was introduced:

$$E^\beta = (E \to cs)$$

For example,

$$cs^\beta = bv$$

Also, we can define the γ dual as

$$E^\gamma = (E \to bs)$$

For the sequence spaces of Fourier coefficients, the Cesàro dual of BK-spaces is important.

$$E^\sigma = (E \to \sigma s)$$

where $\sigma s = \{x \in \omega \mid \frac{s_0+s_1+\cdots+s_n}{n+1}$ converges as $n \to \infty\}$ and $s_n = x_0 + x_1 + \cdots + x_n$.

For example

$$(\widehat{L})^\sigma = \widehat{L^\infty} = \{\widehat{f} \mid f \text{ is essentially bounded on } [-\pi, \pi]\}$$

4. Connection to summability in BK-spaces.

It turns out that summability in BK-spaces, multipliers, duals, and product spaces are closely connected. Here is just one example of how BK-products can be used to find multipliers.

$$(bs \to bv) = (bs\widehat{\otimes}bv^{\gamma})^{\gamma} = (bs\widehat{\otimes}bs)^{\gamma} = (l^{\infty})^{\gamma} = \ell^{1}$$

Here is another result:

A BK-space E has $\sigma K \Leftrightarrow E = q_0 \cdot E$ (i.e., $q_0 \subset (E \to E)$).

Here q_0 is the BK-space of all quasi-convex null sequences. That is, it's the BK-space spanned by the set of convex null sequences.

For example, since $\widehat{L}$ and $\widehat{C}$ have the property σK,

$$\widehat{L} = q_0 \cdot \widehat{L} \quad \text{or} \quad \widehat{L} = q_0\widehat{\otimes}\widehat{L} \quad \text{or} \quad q_0 \subset (\widehat{L} \to \widehat{L})$$

$$\widehat{C} = q_0 \cdot \widehat{C} \quad \text{or} \quad \widehat{C} = q_0\widehat{\otimes}\widehat{C} \quad \text{or} \quad q_0 \subset (\widehat{C} \to \widehat{C})$$

Conclusion.

- I hope that I have shown you that Discrete Functional Analysis is a course that can be taught to advanced undergraduate and first year graduate students.

- Discrete Functional Analysis includes many of the topics of this conference.

- The title Discrete Functionals Analysis is sexy.

- It can bring new blood into our area, increase research, and maybe increase funding.

Postscript: An Example

Theorem of F. and M. Riez:

If $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ are both Fourier series of bounded Radon measures, then they are both Fourier series of an integrable function.

This theorem settled a conjecture of Steinhaus.

We can write the theorem as follows:

$$\widehat{M_c} \cap \widehat{M_s} = \widehat{L}_c \cap \widehat{L}_s$$

By using his theory of the sum of BK-spaces, multipliers, and σ-duality, Günther Goes was able to prove this theorem as follows.

Consider the sum of FK-spaces $E = \widehat{C}_c + \widehat{C}_s$. The topological dual of the sum is the intersection of the topological duals. Furthermore, topological duals can be identified with the σ duals if the spaces have the property σK. This leads to the identity

$$E' = (\widehat{C}_c + \widehat{C}_s)' = (\widehat{C}_c)' \cap (\widehat{C}_s)' = (\widehat{C}_c)^\sigma \cap (\widehat{C}_s)^\sigma = \widehat{M_c} \cap \widehat{M_s}.$$

To show that

$$\widehat{M_c} \cap \widehat{M_s} = \widehat{L}_c \cap \widehat{L}_s$$

it is sufficient to show that $\widehat{M_c} \cap \widehat{M_s}$ has the property σK. This is accomplished by using a theorem that says E' has the property σK if and only if the second dual E'' is σB complete (meaning that $x \in E''$ whenever $\{\sigma^n x\}$ is a bounded subset of E'').

Finding the second dual

$$(\widehat{C}_c + \widehat{C}_s)'' = (\widehat{M_c} \cap \widehat{M_s})' = \widehat{M_c}' + \widehat{M_s}' = \widehat{L}_c^\infty + \widehat{L}_s^\infty$$

and observing that it is indeed σB complete, finishes the proof.

Postscript^2: Open question

For spaces of Fourier coefficient, the norm $\|\widehat{f}\|$ is generally defined in terms of the generating function f, not in terms of the sequence.

(There is one exception, $\widehat{L^2} = \ell^2$.)

For the integrable functions L, can one describe the norm $\|\widehat{f}\| = (a_0, a_1, a_2, a_3, \ldots)$
for $\widehat{f} \in \widehat{L}$ in terms of the coefficients $a_0, a_1, a_2, \ldots$
instead of the generating function

$$\|\widehat{f}\| = \|f\|_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)| dx?$$

Actually, is there a test for when a sequence $x \in \omega$ belongs to $\widehat{L}$?

$$q_0 \subsetneqq \widehat{L} \subsetneqq c_0$$

The same applies to the continuous functions C. Can one describe the norm $\|\widehat{f}\|$ for $\widehat{f} \in \widehat{C}$ in terms of the coefficients $a_0, a_1, a_2, \ldots$ instead of the generating function

$$\|\widehat{f}\| = \|f\|_\infty = \sup_{-\pi \le x \le \pi} |f(x)|?$$

Actually, is there a test for when a sequence $x \in \omega$ belongs to $\widehat{C}$?