

# *A*-Wedge and Weak *A*-Wedge *FK*-Spaces

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## Abstract

The purpose of this paper is to study topological sequence spaces in which the *A*-transform of coordinate vectors (weakly) converge to zero, where *A* is a nonnegative regular factorable matrix. We call these spaces (weak) *A*-wedge *FK*-spaces.

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**Key words:** *FK*-Spaces, Wedge Spaces, *A*-wedge Spaces.

## 1 Introduction

Wedge *FK*-spaces are defined to be topological sequence spaces in which the coordinate vectors converge to zero. Several characterizations of these spaces are given by Bennett in [1]. Ince [3] studied the topological sequence spaces in which the arithmetic means of coordinate vectors converge to zero. Some characterizations of these spaces may be found in [3]. In this paper, we study *A*-wedge *FK*-spaces which are topological sequence spaces in which the *A*-transform of coordinate vectors converge to zero and give some characterizations of these spaces.

In Section 2, we give notation and terminology while in Section 3 we give some preliminary results. Section 4 deals with *A*-wedge *FK*-spaces and Section 5 is devoted to weak *A*-wedge *FK*-spaces.

## 2 Notations and Preliminaries

Let  $w$  denote the space of all real- or complex-valued sequences  $x = (x_n)_{n=1}^{\infty}$ . A *K*-space is a locally convex vector subspace of  $w$  with continuous coordinates. An *FK*-space is a *K*-space which is also a Fréchet space (complete linear metric

space). A  $BK$ -space is a normed  $FK$ -space. The basic properties of such spaces can be found in [6], [7] and [9].

By  $m$ ,  $c$  and  $c_0$  we denote the spaces of all bounded sequences, convergent and null sequences, respectively. These are  $BK$ -spaces under the norm  $\|x\|_\infty = \sup_n |x_n|$ . By  $l^p$ , ( $1 \leq p < \infty$ ), we shall denote the  $BK$ -space of all absolutely  $p$ -summable sequences. As usual,  $l^1$  is denoted simply by  $l$ .

Throughout the paper  $\delta^j$ ,  $j = 1, 2, \dots$ , denotes the sequence  $(0, \dots, 0, 1, 0, \dots)$  with the one in the  $j$ -th position. Let  $\phi := l.hull \{\delta^k : k \in N\}$ . The topological dual of  $X$  is denoted by  $X'$ . A  $K$ -space  $X$  is said to have the property  $AD$  if  $\phi$  is a dense subset of  $X$  and it is said to have the property  $AK$ , if  $X \supset \phi$  and for each  $x \in X$ , we have  $x^{(n)} \rightarrow x$  in  $X$ , where  $x^{(n)} = \sum_{k=1}^n x_k \delta^k = (x_1, x_2, \dots, x_n, 0, \dots)$ .

Let  $X$  be an  $FK$ -space containing  $\phi$ . Following [6],

$$X^f = \{x \in w : x = (f(\delta^1), f(\delta^2), \dots, f(\delta^k), \dots), f \in X'\},$$

$$X^\beta = \left\{ x \in w : \sum_{k=1}^{\infty} x_k y_k \text{ exists for every } y \in X \right\}.$$

Following Bennett [1] we say that a  $K$ -space  $(X, \tau)$  containing  $\phi$  is a wedge space if  $\delta^j \rightarrow 0$  in  $(X, \tau)$  and it is a weak wedge space if  $\delta^j \rightarrow 0$  weakly in  $X$ .

Let  $A = (a_{ij})$  be an infinite matrix. The matrix  $A$  may be considered as a linear transformation  $y = Ax$  of sequences  $x = (x_k)$  by the formula  $y_i = (Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j$ , ( $i = 1, 2, 3, \dots$ ).

For an  $FK$ -space  $X$  generated by seminorms  $\{q_1, q_2, \dots\}$ , we consider the summability domain

$$X_A = \{x \in w : Ax \in X\}.$$

Then  $X_A$  is an  $FK$ -space under the seminorms (see e.g., [7] and [8])

- (i)  $p_i = |x_i|$ , ( $i = 1, 2, \dots$ ),
- (ii)  $h_i(x) = \sup_m \left| \sum_{j=1}^m a_{ij} x_j \right|$ , ( $i = 1, 2, \dots$ ),
- (iii)  $(q_i \circ A)(x) = q_i(Ax)$ , ( $i = 1, 2, \dots$ ).

Throughout the paper  $\{a_n\}$  and  $\{b_n\}$  will be positive real-valued sequences such that

- (i)  $\{a_n\}$  is strictly decreasing with  $\lim_n a_n = 0$ ,
- (ii)  $\{b_n\}$  is nondecreasing with  $\lim_n a_n b_n = 0$  and
- (iii)  $\lim_n a_n \sum_{k=1}^n b_k = 1$ .

In this case, a factorable matrix  $A = (a_{nk})$  defined by  $a_{nk} = a_n b_k$  if  $1 \leq k \leq n$ , and zero otherwise, is a regular matrix; that is,  $c \subset c_A$  and  $\lim_{i \rightarrow \infty} x_i = \lim_{i \rightarrow \infty} (Ax)_i$  for all  $x \in c$ .

For example, the sequences  $\{b_n\}$  and  $\{a_n\}$ , with  $b_n = \binom{n+\alpha}{n}$  and  $a_n = \left(\sum_{k=1}^n b_k\right)^{-1}$  satisfy these conditions. More generally, all Riesz (weighted mean) matrices (see, e.g., Section 3.2 of [2])  $A = (a_{nk})$  with  $a_{nk} = \frac{p_k}{P_n}$  for  $k \leq n$  (and zero otherwise), where  $\{p_n\}$  is positive nondecreasing,  $P_n = \sum_{k=1}^n p_k$ , and  $\frac{p_n}{P_n} \rightarrow 0$  (as  $n \rightarrow \infty$ ), satisfy these conditions.

We define  $A$ -wedge and weak  $A$ -wedge using the matrix  $A$  given above.

**Definition 2.1.** Let  $(X, \tau)$  be a  $K$ -space containing  $\phi$ . Then  $(X, \tau)$  is called an  $A$ -wedge space if the sequence of matrix transformation under the matrix  $A$  of coordinate vectors converges to zero in  $(X, \tau)$ ; that is, the sequence

$$a_n b^{(n)} = a_n \sum_{k=1}^n b_k \delta^k = (a_n b_1, a_n b_2, a_n b_3, \dots, a_n b_n, 0, 0, \dots),$$

converges to zero in  $(X, \tau)$ .

Observe that these are the rows of the matrix  $A$ . Because of the regularity of the factorable matrix  $A$ , every wedge space is an  $A$ -wedge space but the converse doesn't hold. Note that, for  $a_n = \frac{1}{n}$  and  $b_n = 1$ , the  $FK$ -spaces  $c_0, c, m, l^p$ , ( $p > 1$ ) as well as the space of bounded variation  $bv$  are  $A$ -wedge spaces but not wedge spaces.

**Definition 2.2.** A  $K$ -space  $X$  containing  $\phi$  is called a weak  $A$ -wedge space if the sequence

$$a_n b^{(n)} = a_n \sum_{k=1}^n b_k \delta^k = (a_n b_1, a_n b_2, a_n b_3, \dots, a_n b_n, 0, 0, \dots)$$

converges weakly to zero in  $(X, \tau)$ .

Every weak wedge space is a weak  $A$ -wedge space but the converse doesn't hold. For example,  $bv_0 = bv \cap c_0$  is an  $A$ -wedge and weak  $A$ -wedge space but it is neither wedge nor weak wedge.

### 3 Preliminary Results

In this section, we give preliminary results that we need in later sections.

**Theorem 3.1.** (i) A closed subspace, containing  $\phi$ , of an  $A$ -wedge (resp., weak  $A$ -wedge)  $FK$ -space is an  $A$ -wedge (resp., weak  $A$ -wedge)  $FK$ -space.

(ii) An  $FK$ -space which contains an  $A$ -wedge (resp., weak  $A$ -wedge)  $FK$ -space must be an  $A$ -wedge (resp., weak  $A$ -wedge)  $FK$ -space.

(iii) A countable intersection of  $A$ -wedge (resp., weak  $A$ -wedge)  $FK$ -spaces is an  $A$ -wedge (resp., weak  $A$ -wedge)  $FK$ -space.

**Proof.** The elementary properties of  $FK$ -spaces yield the proof (see e.g., Chapter 4 of [6]).■

**Definition 3.1.** Let  $z \in w$ . The space  $V_0^b(z)$  consists of all sequences  $x \in c_0$  for which

$$\sum_{n=1}^{\infty} |z_n| \left| \Delta\left(\frac{x_n}{b_n}\right) \right| < \infty$$

**Lemma 3.2.** Under the norm

$$(3.1) \quad \|x\|_{V_0^b(z)} = \sum_{n=1}^{\infty} |z_n| \left| \Delta\left(\frac{x_n}{b_n}\right) \right| + \sup_n |x_n|$$

$V_0^b(z)$  is an  $FK$ - $AK$  space.

**Proof.** Define the matrix  $V$  by  $v_{nk} = \frac{z_n}{b_k}$  for  $k = n$ ,  $v_{nk} = \frac{-z_n}{b_k}$  for  $k = n+1$ , and  $v_{nk} = 0$  elsewhere. Since  $V_0^b(z) = l_V \cap c_0$ , then  $V_0^b(z)$  is an  $FK$ -space under the norm (3.1) by Theorem 4.3.1 of [6]. For all  $x \in V_0^b(z)$ , we have

$$\|x - x^{(m)}\|_{V_0^b(z)} = \sum_{n=m+1}^{\infty} |z_n| \left| \Delta\left(\frac{x_n}{b_n}\right) \right| + \sup_{n \geq m} |x_n|.$$

Since  $x \in V_0^b(z)$ , then  $\|x - x^{(m)}\|_{V_0^b(z)} \rightarrow 0$  as  $m \rightarrow \infty$ . Hence  $V_0^b(z)$  is an  $AK$ -space.■

**Theorem 3.3.** The space  $V_0^b(z)$  is an  $A$ -wedge space if and only if  $z_n = o(\frac{1}{a_n})$ .

**Proof.** We have for each  $n$ ,

$$\|a_n b^{(n)}\|_{V_0^b(z)} = a_n |z_n| + a_n \sup_{k \leq n} |b_k| = a_n |z_n| + a_n b_n.$$

Since  $\lim_n a_n b_n = 0$ , the sequence  $a_n b^{(n)}$  converges to zero in  $V_0^b(z)$  if and only if  $\lim_n a_n z_n = 0$ .■

**Definition 3.2.** For  $z_n = \frac{1}{a_n}$ , let  $h_a^b = V_0^b(z)$ .

**Lemma 3.4.** The space  $h_a^b$  is an  $FK$ - $AK$  space under the norm

$$(3.2) \quad \|x\|_{h_a^b} = \sum_{n=1}^{\infty} \frac{1}{a_n} \left| \Delta\left(\frac{x_n}{b_n}\right) \right|.$$

**Proof.** By Lemma 3.2, we know that the space  $h_a^b = V_0^b(\frac{1}{a_n})$  is an  $FK$ - $AK$  space under the  $V_0^b(\frac{1}{a_n})$  norm

$$\|x\|_{V_0^b(\frac{1}{a_n})} = \sum_{n=1}^{\infty} \frac{1}{a_n} \left| \Delta\left(\frac{x_n}{b_n}\right) \right| + \sup_n |x_n|.$$

We show that the  $h_a^b$  and  $V_0^b(\frac{1}{a_n})$  norms are equivalent by proving

$$(3.3) \quad \|x\|_{h_a^b} \leq \|x\|_{V_0^b(\frac{1}{a_n})} \leq (1+M) \|x\|_{h_a^b},$$

where  $M = \sup_n a_n b_n$ . The first inequality is obvious. Let  $x \in h_a^b$  and  $n = 1, 2, 3, \dots$ . Since  $\lim_m \frac{x_m}{b_m} = 0$ , we have  $\frac{x_n}{b_n} = \sum_{j=n}^{\infty} \Delta(\frac{x_j}{b_j})$ . Thus

$$\frac{|x_n|}{b_n} = \left| \frac{x_n}{b_n} \right| \leq \sum_{j=n}^{\infty} \left| \Delta(\frac{x_j}{b_j}) \right| = \sum_{j=n}^{\infty} a_j \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right|.$$

Since  $\{a_n\}$  is decreasing,

$$\frac{|x_n|}{b_n} \leq \sum_{j=n}^{\infty} a_n \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right| = a_n \sum_{j=n}^{\infty} \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right|.$$

Thus  $\frac{|x_n|}{a_n b_n} \leq \sum_{j=n}^{\infty} \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right| \leq \sum_{j=1}^{\infty} \frac{1}{a_j} \left| \Delta(\frac{x_j}{b_j}) \right| = \|x\|_{h_a^b}$ , which implies  $\sup_m |x_m| \leq M \sup_m \frac{|x_m|}{a_m b_m} \leq M \|x\|_{h_a^b}$ . This is sufficient to prove the second inequality of (3.3). ■

**Theorem 3.5.** The space  $h_a^b$  is not an  $A$ -wedge space.

**Proof.** Since  $h_a^b = V_0^b(z)$  for  $z_n = \frac{1}{a_n}$ , the result follows from Theorem 3.3. ■

The following spaces relate to duality of  $h_a^b$ .

**Lemma 3.6.** The spaces  $(\sigma_0)_a^b$  and  $(\sigma_\infty)_a^b$  given by

$$(\sigma_0)_a^b = \left\{ x \in w : \lim_n a_n \sum_{k=1}^n b_k x_k = 0 \right\},$$

and

$$(\sigma_\infty)_a^b = \left\{ x \in w : \sup_n a_n \left| \sum_{k=1}^n b_k x_k \right| < \infty \right\}$$

are  $FK$ -spaces under the norm

$$(3.4) \quad \|x\|_{(\sigma_\infty)_a^b} = \sup_n a_n \left| \sum_{k=1}^n b_k x_k \right|.$$

In addition  $(\sigma_0)_a^b$  is an  $AK$ -space under the norm (3.4).

**Proof.** Define the sequence  $Ax$  by  $(Ax)_n = a_n \sum_{k=1}^n b_k x_k$ . Then  $(\sigma_0)_a^b = (c_0)_A$  and  $(\sigma_\infty)_a^b = m_A$ . Since  $A$  is a triangle, then  $(\sigma_0)_a^b$  and  $(\sigma_\infty)_a^b$  are  $FK$ -spaces under the norm (3.4) by Theorem 4.3.12 in [6]. For  $x \in (\sigma_0)_a^b$  and  $\epsilon > 0$ , there exists  $N$  such that  $\sup_n a_n \left| \sum_{k=1}^n b_k x_k \right| < \epsilon$  whenever  $n > N$ . To show that  $(\sigma_0)_a^b$  is

an  $AK$ -space, we show  $\|x - x^{(m)}\|_{(\sigma_\infty)_a^b} = \sup_{n \geq m} a_n \left| \sum_{k=m+1}^n b_k x_k \right| < 2\epsilon$  whenever  $m > N$ . Let  $n \geq m > N$ . Then

$$a_n \left| \sum_{k=m+1}^n b_k x_k \right| \leq a_n \left| \sum_{k=1}^n b_k x_k \right| + a_n \left| \sum_{k=1}^m b_k x_k \right| < \epsilon + a_m \left| \sum_{k=1}^m b_k x_k \right| < 2\epsilon.$$

■

**Lemma 3.7.**  $(h_a^b)^f = (h_a^b)^\beta = (h_a^b)' = (\sigma_\infty)_a^b$ , and  
 $((\sigma_0)_a^b)^f = ((\sigma_0)_a^b)^\beta = ((\sigma_0)_a^b)' = h_a^b$ .

**Proof.** Since  $h_a^b$  is an  $AK$ -space, then  $(h_a^b)^f = (h_a^b)^\beta = (h_a^b)'$  by Theorems 7.2.7 and 7.2.12 of [6]. Let  $x \in (\sigma_\infty)_a^b$  and  $y \in h_a^b$ . Because of  $y \in h_a^b$ , we have  $\lim_n \frac{y_n}{a_n b_n} = 0$ . By Abel summation

$$\sum_{k=m}^n x_k y_k = \sum_{k=m}^{n-1} a_k s_k \frac{1}{a_k} \Delta \left( \frac{y_k}{b_k} \right) + a_n s_n \frac{y_n}{a_n b_n} - a_{m-1} s_{m-1} \frac{y_m}{a_{m-1} b_m},$$

where  $s_k = \sum_{j=1}^k b_j x_j$ . Since  $\|x\|_{(\sigma_\infty)_a^b} = \sup_k a_k |s_k|$ , we have, as  $m, n \rightarrow \infty$ ,

$$\left| \sum_{k=m}^n x_k y_k \right| \leq \|x\|_{(\sigma_\infty)_a^b} \left( \sum_{k=m}^{n-1} \frac{1}{a_k} \left| \Delta \left( \frac{y_k}{b_k} \right) \right| + \left| \frac{y_n}{a_n b_n} \right| + \left| \frac{y_m}{a_{m-1} b_m} \right| \right) \rightarrow 0.$$

Hence  $(\sigma_\infty)_a^b \subset (h_a^b)^\beta$ .

Conversely let  $u \in (h_a^b)^f$ . That is,  $u_k = f(\delta^k)$ , where  $f \in (h_a^b)'$ . Then

$$\begin{aligned} a_n \left| \sum_{k=1}^n b_k u_k \right| &= \left| a_n \sum_{k=1}^n b_k f(\delta^k) \right| = \left| f \left( a_n \sum_{k=1}^n b_k \delta^k \right) \right| \\ &\leq \|f\| \left\| a_n b^{(n)} \right\|_{h_a^b} = \|f\|. \end{aligned}$$

So  $u \in (\sigma_\infty)_a^b$ .

For the second part, since  $h_a^b$  is an  $FK$ - $AK$ -space, we have

$$h_a^b \subset ((\sigma_0)_a^b)^f = ((\sigma_0)_a^b)^\beta = ((\sigma_0)_a^b)'$$

The reverse inclusion follows from Corollary 1(iii) of [5]. According to that Corollary,

$$((\sigma_0)_a^b)^\beta = \left\{ x \in w : \sum_{n=1}^{\infty} \frac{1}{a_n} \left| \Delta \left( \frac{x_n}{b_n} \right) \right| < \infty, \sup \left| \frac{x_n}{a_n b_n} \right| < \infty \right\}.$$

Since we assume  $\lim_n a_n b_n = 0$ , we have

$$((\sigma_0)_a^b)^\beta \subset \left\{ x \in w : \sum_{n=1}^{\infty} \frac{1}{a_n} \left| \Delta \left( \frac{x_n}{b_n} \right) \right| < \infty, \lim |x_n| = 0 \right\} = h_a^b.$$

■

**Definition 3.3.** Let  $s = \{s_n\}_{n=1}^{\infty}$  always denote a strictly increasing sequence of integers with  $s_1 = 0$ . We shall be interested in spaces of the form:

$$c_a^b |s| = \left\{ x \in w : \lim_n x_n = 0 \text{ and } \sup_n \sum_{j=s_n+1}^{s_{n+1}} \frac{1}{a_j} \left| \Delta\left(\frac{x_j}{b_j}\right) \right| < \infty \right\}.$$

**Lemma 3.8.** For any strictly increasing sequence  $s = \{s_n\}_{n=1}^{\infty}$ , the space  $c_a^b |s|$  is an *FK*-space with the norm

$$(3.5) \quad \|x\|_{c_a^b |s|} = \sup_n \sum_{j=s_n+1}^{s_{n+1}} \frac{1}{a_j} \left| \Delta\left(\frac{x_j}{b_j}\right) \right|.$$

Furthermore,

$$h_a^b \subset c_a^b |s| \subset c_0 \subset m.$$

**Proof.** The proof is essentially the same as that of Lemma 3.2. ■

Similarly, the proof of the following is like that of Theorem 3.3.

**Theorem 3.9.** For any strictly increasing sequence  $s = \{s_n\}_{n=1}^{\infty}$ , the space  $c_a^b |s|$  is not an *A*-wedge space.

**Lemma 3.10.** Assume that  $\lim_j a_j z_j^n = 0$ , ( $n = 1, 2, 3, \dots$ ). Then there exists  $z \in w$  with  $\lim_j a_j z_j = 0$  such that  $\lim_j \frac{z_j^n}{z_j} = 0$ , ( $n = 1, 2, 3, \dots$ ). Moreover, for any such  $z$ , we have  $V_0^b(z) \subset \bigcap_{n=1}^{\infty} V_0^b(z^n)$ .

**Proof.** Since  $\lim_j a_j z_j^n = 0$ , ( $n = 1, 2, 3, \dots$ ), we may choose a sequence  $\{j_k\}_{k=1}^{\infty}$  of positive integers such that

$$1 = j_0 < j_1 < j_2 < \dots < j_k < \dots$$

and

$$\max_{1 \leq n \leq k} |a_j z_j^n| < \frac{1}{4^k}, \quad (j \geq j_k ; k = 1, 2, \dots).$$

Define  $z \in w$  as follows :

$$z_j = \frac{1}{2^k a_j}, \quad (j_k \leq j < j_{k+1}; k = 0, 1, 2, \dots).$$

Then  $\lim_j a_j z_j = 0$  and, fixing  $n$ , we get

$$(3.6) \quad \left| \frac{z_j^n}{z_j} \right| = \left| \frac{a_j z_j^n}{a_j z_j} \right| < \frac{1}{2^k} \text{ whenever } j_k \leq j < j_{k+1} \text{ and } k \geq n.$$

Thus  $\lim_j \frac{z_j^n}{z_j} = 0$  for each  $n$ . The second part of the proof follows from inequality (3.6). ■

**Remark 3.1.** In Lemma 3.10 above, we may choose the sequence  $\{j_k\}$  of positive integers such that

$$a_j b_j < \frac{1}{2^k} \text{ for } j \geq j_k$$

and

$$a_{j_{k+1}} < \frac{1}{2} a_{j_k} \text{ for } k = 0, 1, 2, \dots$$

Then

$$z_j = \frac{1}{2^k a_j} \text{ for } j_k \leq j < j_{k+1}; \quad k = 0, 1, 2, \dots$$

and

$$\frac{z_j}{b_j} = \frac{a_j z_j}{a_j b_j} > 1 \text{ for } j_k \leq j < j_{k+1}; \quad k = 0, 1, 2, \dots$$

**Lemma 3.11.** Assume that  $\lim_j a_j z_j^0 = 0$  for some  $z^0 \in w$ . Then there exists  $z' \in w$  with  $\lim_j a_j z_j' = 0$  such that  $V_0^b(z') \subset V_0^b(z^0)$  and  $V_0^b(z')$  is an  $FK$ -space under the norm

$$\|x\|' = \sum_{j=1}^{\infty} z_j' \left| \Delta\left(\frac{x_j}{b_j}\right) \right|.$$

**Proof.** Let  $z^n = z^0$  for  $n = 1, 2, 3, \dots$  and assume the sequence  $\{j_k\}$  of positive integers is chosen according to the proof of Lemma 3.10 and Remark 3.1 above. Then, by Lemma 3.10, there exists a sequence  $z' \in w$  with  $\lim_j a_j z_j' = 0$  such that  $V_0^b(z') \subset V_0^b(z^0)$ . It is sufficient to show that, for all  $x \in V_0^b(z')$ , we have

$$(3.7) \quad \sup_n |x_n| \leq 2 \|x\|'.$$

Although the sequence  $z_j'$  may not be increasing for all  $j = 1, 2, 3, \dots$ , it has an increasing sawtooth shape as observed by (3.8), (3.9), and (3.10) below:

Since  $\{a_j\}$  is strictly decreasing, for each  $i = 0, 1, 2, \dots$

$$(3.8) \quad z_j' = \frac{1}{2^i a_j} \text{ is strictly increasing for } j_i \leq j < j_{i+1}$$

and for each  $j_i \leq n < j_{i+1}$ ,

$$(3.9) \quad z_{j_i}' = \frac{1}{2^i a_{j_i}} \leq \frac{1}{2^i a_n} = z_n' < \frac{1}{2^i a_{j_{i+1}}} = \frac{2}{2^{i+1} a_{j_{i+1}}} = 2z_{j_{i+1}}'.$$

Also  $\{z_{j_i}'\}_{i=0}^{\infty}$  is positive and strictly increasing because  $a_{j_{i+1}} < \frac{1}{2} a_{j_i}$ ; that is,

$$(3.10) \quad z_{j_i}' = \frac{1}{2^i a_{j_i}} < \frac{1}{2^{i+1} a_{j_{i+1}}} = z_{j_{i+1}}'.$$

Let  $x \in V_0^b(z')$  and  $j_k \leq n < j_{k+1}$ . Define  $\Delta_i = \frac{x_{j_i}}{b_{j_i}} - \frac{x_{j_{i+1}}}{b_{j_{i+1}}}$ . ( $i = 0, 1, 2, \dots$ ). Since  $\lim_j \frac{x_j}{b_j} = 0$ , and hence  $\lim_i \frac{x_{j_i}}{b_{j_i}} = 0$ , we have

$$\frac{x_n}{b_n} = \sum_{j=n}^{j_{k+1}-1} \Delta\left(\frac{x_j}{b_j}\right) + \sum_{i=k+1}^{\infty} \Delta_i.$$

Thus

$$(3.11) \quad \left| \frac{x_n}{b_n} \right| \leq \sum_{j=n}^{j_{k+1}-1} \left| \Delta\left(\frac{x_j}{b_j}\right) \right| + \sum_{i=k+1}^{\infty} |\Delta_i|.$$

By (3.8) and (3.10), we have

$$(3.12) \quad \left| \frac{x_n}{b_n} \right| \leq \frac{1}{z'_n} \sum_{j=n}^{j_{k+1}-1} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right| + \frac{1}{z'_{j_{k+1}}} \sum_{i=k+1}^{\infty} z'_{j_i} |\Delta_i|.$$

By (3.9), we have

$$\left| \frac{x_n}{b_n} \right| \leq \frac{1}{z'_n} \sum_{j=n}^{j_{k+1}-1} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right| + \frac{2}{z'_n} \sum_{i=k+1}^{\infty} z'_{j_i} |\Delta_i|.$$

By Remark 3.1, we have  $\frac{z'_n}{b_n} > 1$ . Thus

$$(3.13) \quad |x_n| \leq \frac{z'_n |x_n|}{b_n} \leq \sum_{j=n}^{j_{k+1}-1} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right| + 2 \sum_{i=k+1}^{\infty} z'_{j_i} |\Delta_i|.$$

Expanding  $z'_{j_i} |\Delta_i|$  in (3.13), we have by (3.8),

$$z'_{j_i} |\Delta_i| = z'_{j_i} \left| \frac{x_{j_i}}{b_{j_i}} - \frac{x_{j_{i+1}}}{b_{j_{i+1}}} \right| \leq \sum_{j=j_i}^{j_{i+1}-1} z'_{j_i} \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \leq \sum_{j=j_i}^{j_{i+1}-1} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right|.$$

Inserting this into inequality (3.13),

$$|x_n| \leq \sum_{j=n}^{j_{k+1}-1} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right| + 2 \sum_{i=k+1}^{\infty} \sum_{j=j_i}^{j_{i+1}-1} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \leq 2 \sum_{j=n}^{\infty} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right|,$$

gives us the desired result for all  $n$

$$|x_n| \leq 2 \sum_{j=n}^{\infty} z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \leq 2 \|x\|'.$$

■

## 4 $A$ -Wedge $FK$ -Spaces

In this section we study  $A$ -wedge  $FK$ -spaces. First we give the fundamental characterization of  $A$ -wedge spaces.

**Theorem 4.1.** The following conditions are equivalent for an  $FK$ -space  $(X, \tau)$  :

- (i)  $X$  is an  $A$ -wedge space,
- (ii)  $X$  contains  $V_0^b(z)$  for some  $z \in w$  such that  $z_j = o(\frac{1}{a_j})$ ,
- (iii)  $X$  contains  $c_a^b |s|$  for some strictly increasing sequence  $s$  and the inclusion mapping  $I : (c_a^b |s|, \|\cdot\|_{c_a^b |s|}) \rightarrow (X, \tau)$  is compact,
- (iv)  $X$  contains  $h_a^b$  and the inclusion mapping  $I : (h_a^b, \|\cdot\|_{h_a^b}) \rightarrow (X, \tau)$  is compact.

**Proof.** (ii)  $\implies$  (i) follows from Theorems 3.1 and 3.3.

(i)  $\implies$  (ii). Let  $\{q_n\}_{n=1}^\infty$  be a defining family of seminorms for the topology  $\tau$  and let

$$z_j^n := q_n(b^{(j)}) = q_n\left(\sum_{i=1}^j b_i \delta^i\right); \quad j, n = 1, 2, \dots$$

Suppose  $x \in \bigcap_{n=1}^\infty V_0^b(z^n)$ . Then  $x \in c_0$  and for all  $n = 1, 2, 3, \dots$  we have

$$\sum_{j=1}^\infty |z_j^n| \left| \Delta\left(\frac{x_j}{b_j}\right) \right| = \sum_{j=1}^\infty q_n(b^{(j)}) \left| \Delta\left(\frac{x_j}{b_j}\right) \right| = \sum_{j=1}^\infty q_n\left(\left(\Delta\left(\frac{x_j}{b_j}\right)\right)b^{(j)}\right) < \infty.$$

Since  $X$  is complete,  $\sum_{j=1}^\infty \left(\Delta\left(\frac{x_j}{b_j}\right)\right)b^{(j)}$  converges in  $(X, \tau)$ . Coordinatewise, it converges to  $x$ , since, for  $k \geq i$ , the  $i^{\text{th}}$  coordinate of  $\sum_{j=1}^k \left(\Delta\left(\frac{x_j}{b_j}\right)\right)b^{(j)}$  is  $x_i - \frac{b_i}{b_{k+1}} x_{k+1}$ ,

which tends to  $x_i$  as  $k \rightarrow \infty$ . Since  $X$  is a  $K$ -space,  $x = \sum_{j=1}^\infty \left(\Delta\left(\frac{x_j}{b_j}\right)\right)b^{(j)} \in X$ .

So  $\bigcap_{n=1}^\infty V_0^b(z^n) \subset X$ . Since  $X$  is an  $A$ -wedge space,  $\lim_j q_n(a_n b^{(n)}) = \lim_j a_j z_j^n = 0$ , ( $n = 1, 2, 3, \dots$ ). By Lemma 3.10 we may choose  $z \in w$  such that  $\lim_j a_j z_j = 0$

and  $V_0^b(z) \subset \bigcap_{n=1}^\infty V_0^b(z^n) \subset X$ .

(ii)  $\implies$  (iii). Let  $V_0^b(z) \subset X$  for some  $z$  with  $z_j = o(\frac{1}{a_j})$ . By Lemma 3.11,  $V_0^b(z)$  has a subset  $V_0^b(z')$  with norm  $\|x\|' = \sum_{j=1}^\infty z'_j \left| \Delta\left(\frac{x_j}{b_j}\right) \right|$ . Take  $s_0 = 0$  and  $\{s_n\}_{n=1}^\infty$  denotes a strictly increasing sequence of integers satisfying,

$$(4.1) \quad |a_j z'_j| \leq \frac{1}{2^n}, \quad j \geq s_n; \quad (n = 1, 2, 3, \dots).$$

Let  $x \in c_a^b |s|$ . Suppose  $m, p \in N$ ,  $m \leq p$ . Then by (4.1) we get

$$\sum_{j=s_{m+1}}^{s_{p+1}} |z'_j| \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \leq \sum_{n=m}^p \sum_{j=s_{n+1}}^{s_{n+1}} a_j |z'_j| \frac{1}{a_j} \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \leq \|x\|_{c_a^b |s|} \sum_{n=m}^p \frac{1}{2^n} \rightarrow 0$$

Thus  $x \in V_0^b(z')$  and  $c_a^b |s| \subset V_0^b(z') \subset X$ .

Now assume that  $U \subset c_a^b |s|$  be such that  $\|x\|_{c_a^b |s|} \leq M$  for all  $x \in U$ . It is clear that  $U \subset V_0^b(z')$ . For  $s_n \leq m < s_{n+1}$  and  $x \in U$ , by (4.1) we get

$$\begin{aligned} \|x - x^{(m)}\|' &= \sum_{j=m+1}^{\infty} |z'_j| \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \leq \sum_{i=n}^{\infty} \sum_{j=s_{i+1}}^{s_{i+1}} a_j |z'_j| \frac{1}{a_j} \left| \Delta\left(\frac{x_j}{b_j}\right) \right| \\ &\leq \|x\|_{c_a^b |s|} \sum_{i=m}^{\infty} \frac{1}{2^i} \rightarrow 0 \text{ (uniformly on } U), (m \rightarrow \infty). \end{aligned}$$

Hence  $x^{(m)} \rightarrow x$ ,  $(m \rightarrow \infty)$ , in  $(V_0^b(z'), \|x\|')$  uniformly on  $U$ . Since  $V_0^b(z')$  is an  $AK$ -space, then by Lemma 2 of [1],  $U$  is relatively compact in  $V_0^b(z')$ . Since the inclusion mapping  $I : V_0^b(z') \rightarrow X$  is continuous,  $I(U) = U$  is relatively compact in  $X$ . Thus the inclusion mapping  $I : (c_a^b |s|, \|\cdot\|_{c_a^b |s|}) \rightarrow (X, \tau)$  is compact.

(iii)  $\implies$  (iv). Because the inclusion mapping  $I : h_a^b \rightarrow c_a^b |s|$  is continuous, the proof is trivial.

(iv)  $\implies$  (i). Since  $\|a_n b^{(n)}\|_{h_a^b} = 1$ , the set  $B = \{a_n b^{(n)} : n = 1, 2, \dots\}$  is a bounded subset of  $h_a^b$ . In addition since the inclusion mapping  $I : (h_a^b, \|\cdot\|_{h_a^b}) \rightarrow (X, \tau)$  is compact,  $I(B) = B$  is  $\tau$ -relatively compact in  $X$ . Hence, by Theorem 2.3.11 of [4], since  $a_n b^{(n)} \rightarrow 0$  in  $w$ , we have  $a_n b^{(n)} \rightarrow 0$  in  $(X, \tau)$ . ■

**Theorem 4.2.** Suppose  $z \in (\sigma_0)_a^b$ . Then  $z^\beta := \left\{ x \in w : \sum_{k=1}^{\infty} z_k x_k \text{ converges} \right\}$  is an  $A$ -wedge  $FK$ -space.

**Proof.** The space  $z^\beta$  is an  $FK$ -space with seminorms  $p_i(x) = |x_i|$ ,  $(i = 1, 2, \dots)$ , and  $P_0(x) = \sup_m \left| \sum_{k=1}^m z_k x_k \right|$ , by Theorem 4.3.7 in [6]. Let  $z \in (\sigma_0)_a^b$ . Since, for each  $i = 1, 2, 3, \dots$ ,

$$p_i(a_n b^{(n)}) = \begin{cases} a_n b_i & , \text{ if } i \leq n \\ 0 & , \text{ if } i > n \end{cases} \leq a_n b_n \rightarrow 0, (n \rightarrow \infty),$$

it remains to show that  $P_0(a_n b^{(n)}) = \max_{1 \leq m \leq n} a_n \left| \sum_{k=1}^m z_k b_k \right| \rightarrow 0, (n \rightarrow \infty)$ . Since the sequence  $\{a_n\}$  is decreasing and  $\lim_n a_n = 0$ , we choose a sequence  $\{\zeta_N\}$  of natural numbers for which  $\frac{a_{\zeta_N-1}}{a_{\zeta_N}} \geq 2^N$  and  $a_\zeta \left| \sum_{i=1}^{\zeta} z_i b_i \right| \leq 2^{-N}, (\forall \zeta \geq \zeta_N)$ .

Then for any  $N > 2$ , take  $n \geq \zeta_N$ . We have

$$(i) \quad a_n \left| \sum_{k=1}^m z_k b_k \right| = \frac{a_n}{a_m} a_m \left| \sum_{k=1}^m z_k b_k \right| \leq 2^{-(N-1)}, \text{ for } \zeta_{N-1} \leq m \leq n$$

$$(ii) \quad a_n \left| \sum_{k=1}^m z_k b_k \right| \leq \frac{a_{\zeta_N}}{a_{\zeta_{N-1}}} a_m \left| \sum_{k=1}^m z_k b_k \right| \leq 2^{-N} \sup_m a_m \left| \sum_{k=1}^m z_k b_k \right|, \text{ for } m <$$

$\zeta_{N-1}$ .

Hence

$$P_0(a_n b^{(n)}) = \max \left\{ 2^{-(N-1)}, 2^{-N} \sup_{m < \zeta_{N-1}} a_m \left| \sum_{k=1}^m z_k b_k \right| \right\}$$

which tends to zero as  $n \rightarrow \infty$ . ■

Now we give the following result.

**Corollary 4.3.** The intersection of all  $A$ -wedge  $FK$ -spaces is  $h_a^b$ .

**Proof.** Let the intersection of all  $A$ -wedge  $FK$ -spaces be  $Y$ . By Theorem 4.1 (i)  $\implies$  (iv), Theorem 4.2, and Lemma 3.7, we have

$$h_a^b \subset Y \subset \bigcap \{ z^\beta : z \in (\sigma_0)_a^b \} = \{ (\sigma_0)_a^b \}^\beta = h_a^b.$$

Hence the result obtained. ■

**Corollary 4.4.**  $\bigcap_{z_n = o(\frac{1}{a_n})} V_0^b(z) = h_a^b$ .

**Proof.** By Theorem 3.3, if  $z_n = o(\frac{1}{a_n})$ , then  $V_0^b(z)$  is an  $A$ -wedge  $FK$ -space. Also by Theorem 4.1 (iv),  $h_a^b \subset \bigcap_{z_n = o(\frac{1}{a_n})} V_0^b(z)$ . The reverse inclusion is obtained by Theorem 4.1 (ii) and Corollary 4.3. ■

**Remark 4.1.** By Corollary 4.3, there is no smallest  $A$ -wedge space.

## 5 Weak $A$ -Wedge $FK$ -Spaces

In this section, we deal with weak  $A$ -wedge  $FK$ -spaces

**Theorem 5.1.** An  $FK$ -space  $(X, \tau)$  is a weak  $A$ -wedge space if and only if  $X$  contains  $h_a^b$  and the inclusion mapping  $I : (h_a^b, \|\cdot\|_{h_a^b}) \rightarrow (X, \tau)$  is weakly compact.

**Proof. Necessity:** Let  $(X, \tau)$  be a weak  $A$ -wedge space. Then for all  $f \in X'$ ,

$$(5.1) \quad f(a_n b^{(n)}) = f\left(a_n \sum_{k=1}^n b_k \delta^k\right) = a_n \sum_{k=1}^n b_k f(\delta^k) \rightarrow 0, (n \rightarrow \infty),$$

and thereby  $\{f(\delta^k)\} \in (\sigma_\infty)_a^b$ . Thus  $X^f \subset (\sigma_\infty)_a^b$ . Since  $(\sigma_\infty)_a^b = (h_a^b)^f$  and  $h_a^b$  is an  $AD$ -space, then  $h_a^b \subset X$  by Theorem 8.6.1 in [6]. This inclusion requires

that the inclusion mapping  $I : h_a^b \rightarrow X$  is continuous. Because  $h_a^b$  is an  $AK$ -space, we have for all  $x \in h_a^b$  and  $f \in X'$  that

$$f \left( \sum_{k=1}^{\infty} x_k \delta^k \right) = \sum_{k=1}^{\infty} x_k f(\delta^k) = \langle I(x), f \rangle = \langle x, f(\delta^k) \rangle.$$

On the other hand,  $\{f(\delta^k)\} \in (\sigma_0)_a^b$  for all  $f \in X'$  by (5.1).

Thus, since  $\sigma((\sigma_0)_a^b)', (\sigma_0)_a^b = \sigma(h_a^b, (\sigma_0)_a^b)$ , then the mapping  $I : (h_a^b, \sigma(h_a^b, (\sigma_0)_a^b)) \rightarrow (X, \sigma(X, X'))$  is continuous. By the Banach-Alaoglu Theorem (Theorem 1, Section 13.3 of [7]), the set  $B = \{x \in h_a^b : \|\cdot\|_{h_a^b} \leq 1\}$  is  $\sigma(h_a^b, (\sigma_0)_a^b)$ -compact and hence  $I(B) = B$  is  $\sigma(X, X')$ -compact. Consequently the inclusion mapping  $I : (h_a^b, \|\cdot\|_{h_a^b}) \rightarrow (X, \tau)$  is weakly compact.

**Sufficiency:** Let  $h_a^b \subset X$  and the inclusion mapping  $I : (h_a^b, \|\cdot\|_{h_a^b}) \rightarrow (X, \tau)$  be weakly compact. Then  $B = \{x \in h_a^b : \|\cdot\|_{h_a^b} \leq 1\}$  is  $\sigma(X, X')$ -relatively compact. Hence, by Theorem 2.3.11 of [4],  $a_n b^{(n)} \rightarrow 0, (n \rightarrow \infty)$ , in  $\sigma(X, X')$  since it converges to zero in  $w$ . ■

Now we have the following

**Corollary 5.2.** The intersection of all weak  $A$ -wedge  $FK$ -spaces is  $h_a^b$ .

**Proof.** The proof is like that of Corollary 4.3 by using Theorems 5.1 and 4.2. ■

Using Theorem 3.1 for weak  $A$ -wedge  $FK$ -spaces, we obtain the following.

**Remark 5.1.** There is no smallest weak  $A$ -wedge  $FK$ -space.

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