FORWARD COMPACTNESS

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ABSTRACT. A real function f is continuous if and only if $(f(x_n))$ is a convergent sequence whenever (x_n) is convergent and a subset E of $\mathbf{R}$ is compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a convergent subsequence whose limit is in E where $\mathbf{R}$ is the set of real numbers. These well known results suggest us to introduce a concept of forward continuity in the sense that a function f is forward continuous if $\lim_{n\to\infty} \Delta f(x_n) = 0$ whenever $\lim_{n\to\infty} \Delta x_n = 0$ and a concept of forward compactness in the sense that a subset E of $\mathbf{R}$ is forward compact if any sequence $\mathbf{x} = (x_n)$ of points in E has a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of the sequence $\mathbf{x}$ such that $\lim_{k\to\infty} \Delta z_k = 0$ where $\Delta y_k = y_{k+1} - y_k$. We prove that any forward continuous function defined on a forward compact subset of $\mathbf{R}$ is uniformly continuous, uniform limit of forward continuous functions is forward continuous, any forward continuous function is continuous. Some other results related to forward continuity and continuity are also obtained and some open problems are discussed.

1. INTRODUCTION

A real function f is continuous if and only if $(f(x_n))$ is a convergent sequence whenever (x_n) is convergent. Regardless of limit, this is equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is. Using the idea of continuity of a real function in terms of sequences, we might introduce a concept of forward continuity in the sense that a function f is forward continuous if it transforms forward convergent to 0 sequences to forward convergent to 0 sequences, i.e. $(f(x_n))$ is forward convergent to 0 whenever (x_n) is forward convergent to 0. Before we begin, some definitions and notation will be given in the following. Throughout this paper, $\mathbf{N}$ will denote the set of all positive integers. We will use boldface letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, ... for sequences $\mathbf{x} = (x_n)$, $\mathbf{y} = (y_n)$, $\mathbf{z} = (z_n)$, ... of terms in $\mathbf{R}$. c and Δ will denote the set of all convergent sequences and the set of all forward convergent to 0 sequences of points in $\mathbf{R}$ where a sequence $\mathbf{x} = (x_n)$ is called forward convergent to 0 if $\lim_{n\to\infty} \Delta x_n = 0$.

Following the idea given in a 1946 American Mathematical Monthly problem [4], a number of authors Posner [12], Iwinski [10], Srinivasan [16], Antoni [1], Antoni and Salat [2], Spigel and Krupnik [15] have studied A-continuity defined by a regular summability matrix A. Some authors, Öztürk [11], Savaş and Das [13], Borsik and Salat [3]) have studied A-continuity for methods of almost convergence or for related methods.

Fast [8] introduced the definition of statistical convergence. Recall that for a subset M of $\mathbf{N}$ the asymptotic density of M, denoted by $\delta(M)$, is given by

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Date: September 17, 2009.

2000 *Mathematics Subject Classification.* Primary: 40A05; Secondary: 26A05.

Key words and phrases. Sequences, series, summability, compactness,continuity.

$$\delta(M) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in M\}|,$$

if this limit exists, where $|\{k \leq n : k \in M\}|$ denotes the cardinality of the set $\{k \leq n : k \in M\}$. A sequence (x_n) is statistically convergent to ℓ if

$$\delta(\{n : |x_n - \ell| > \epsilon\}) = 0,$$

for every $\epsilon > 0$. In this case ℓ is called the statistical limit of $\mathbf{x}$. $st(\mathbf{R})$ will denote the set of statistically convergent sequences. Schoenberg [14] studied some basic properties of statistical convergence and also studied the statistical convergence as a summability method. Fridy [9] gave charecterizations of statistical convergence.

Recently, Connor and Grosse-Erdman [5] have given sequential definitions of continuity for real functions calling G-continuity instead of A-continuity and their results covers the earlier works related to A-continuity where a method of sequential convergence, or briefly a method, is a linear function G defined on a linear subspace of s, denoted by c_G, into $\mathbf{R}$. A sequence $\mathbf{x} = (x_n)$ is said to be G-convergent to ℓ if $\mathbf{x} \in c_G$ and $G(\mathbf{x}) = \ell$. In particular, lim denotes the limit function $\lim \mathbf{x} = \lim_n x_n$ on the linear space c and $st - \lim$ denotes the statistical limit function $st - \lim \mathbf{x} = st - \lim_n x_n$ on the linear space $st(\mathbf{R})$. A function f is called G-continuous at a point u provided that whenever a sequence $\mathbf{x} = (x_n)$ of terms in the domain of f is G-convergent to u, then the sequence $f(\mathbf{x}) = (f(x_n))$ is G-convergent to $f(u)$. A method G is called regular if every convergent sequence $\mathbf{x} = (x_n)$ is G-convergent with $G(\mathbf{x}) = \lim \mathbf{x}$. A method is called subsequential if whenever $\mathbf{x}$ is G-convergent with $G(\mathbf{x}) = \ell$, then there is a subsequence (x_{n_k}) of $\mathbf{x}$ with $\lim_k x_{n_k} = \ell$.

The purpose of this note is to introduce a concept of forward continuity of a function and a concept of forward compactness of a subset of $\mathbf{R}$ which cannot be given by means of any G and that forward continuity implies the ordinary continuity and is implied by uniform continuity and to prove that any forward continuous function on a forward compact subset E of $\mathbf{R}$ is uniformly continuous and that uniform limit of a sequence of forward continuous functions is forward continuous.

2. FORWARD COMPACTNESS

We say that a sequence $\mathbf{x} = (x_n)$ is forward convergent to a number ℓ if $\lim_{k\to\infty} \Delta x_k = \ell$ where $\Delta x_k = x_{k+1} - x_k$. Now we give the definition of forward compactness of a subset of $\mathbf{R}$.

Definition 1. A subset E of $\mathbf{R}$ is called forward compact if whenever $\mathbf{x} = (x_n)$ is a sequence of points in E there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ with $\lim_{k\to\infty} \Delta z_k = 0$.

Firstly, we note that any finite subset of $\mathbf{R}$ is forward compact, union of two forward compact subsets of $\mathbf{R}$ is forward compact and intersection of any forward compact subsets of $\mathbf{R}$ is forward compact. Furthermore any subset of a forward compact set is forward compact and any bounded subset of $\mathbf{R}$ is forward compact. Any compact subset of $\mathbf{R}$ is also forward compact and the converse is not always true and there are forward compact subsets of $\mathbf{R}$ which are unbounded. For example the set $K = \{\sqrt{n} : n \in \mathbf{N}\}$ is forward compact, but it is not compact. On the other hand, the set $\mathbf{N}$ is not forward compact. We note that any slowly oscillating compact subset of $\mathbf{R}$ is forward compact (see [7] for the definition of slowly oscillating compactness).

We note that this definition of forward compactness can not be obtained by any G-sequential compactness, i.e. by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $a_{nk} = -1$ if $k = n$ and $a_{kn} = 1$ if $k = n+1$ and

$$G(x) = \lim A\mathbf{x} = \lim_{k\to\infty} \sum_{n=1}^{\infty} a_{kn}x_n = \lim_{k\to\infty} \Delta x_k \qquad (*)$$

(see [6] for the definition of G-sequential compactness). Despite that G-sequential compact subsets of $\mathbf{R}$ should include the singleton set $\{0\}$, forward compact subsets of $\mathbf{R}$ do not have to include the singleton $\{0\}$.

A real function f is continuous if and only if, for each point x_0 in the domain, $\lim_{n\to\infty} f(x_n) = f(x_0)$ whenever $\lim_{n\to\infty} x_n = x_0$. This is equivalent to the statement that $(f(x_n))$ is a convergent sequence whenever (x_n) is. This is also equivalent to the statement that $(f(x_n))$ is a Cauchy sequence whenever (x_n) is Cauchy. These well known results for continuity for real functions in terms of sequences might suggest us to give a new type continuity, namely, forward continuity:

Definition 2. A function f is called forward continuous on E if the sequence $f(\mathbf{x}) = (f(x_n))$ is forward convergent to 0 whenever $\mathbf{x} = (x_n)$ is a sequence of terms in E which is forward convergent to 0 .

We note that this definition of continuity can not be obtained by any A-continuity, i.e. by any summability matrix A, even by the summability matrix $A = (a_{nk})$ defined by $(*)$ however for this special summability matrix A if A-continuity of f at the point 0 implies forward continuity of f, then $f(0) = 0$; and if forward continuity of f implies A-continuity of f at the point 0, then $f(0) = 0$.

We also note that sum of two forward continuous functions is forward continuous and composite of two forward continuous functions is forward continuous but the product of two forward continuous functions need not be forward continuous as it can be seen by considering product of the forward continuous function $f(x) = x$ with itself.

We note that if f and g are forward continuous functions, then so are $max\{f, g\}$ and $min\{f, g\}$. More generally, if (f_n) is a sequence of forward continuous functions, then so are $supf_n$ and $inff_n$.

In connection with forward convergent to 0 sequences and convergent sequences the problem arises to investigate the following types of continuity of functions on $\mathbf{R}$.

(δ): $(x_n) \in \Delta \Rightarrow (f(x_n)) \in \Delta$
(δc): $(x_n) \in \Delta \Rightarrow (f(x_n)) \in c$
(c): $(x_n) \in c \Rightarrow (f(x_n)) \in c$
(d): $(x_n) \in c \Rightarrow (f(x_n)) \in \Delta$

We see that (δ) is forward continuity of f and (c) states the ordinary continuity of f. It is easy to see that (δc) implies (δ), and (δ) does not imply (δc); and (δ) implies (d), and (d) does not imply (δ); (δc) implies (c) and (c) does not imply (δc); and (c) is eqivalent to (d).

Since statistical continuity is equivalent to ordinary continuity, (c) can be replaced by statistical continuity, i.e. $st - \lim_{n\to\infty} f(x_n) = f(\ell)$ whenever $\mathbf{x} = (x_n)$ is a statisctically convergent sequence with $st - \lim_{n\to\infty} x_n) = \ell$. More generally

(c) can be replaced by G-sequential continuity of f for any regular subsequential method G (see Corollary to Theorem 5 on page 106 of [5]).

Now we give the implication (δ) implies (c), i.e. any forward continuous function is continuous in the ordinary sense.

Theorem 1 If f is forward continuous on a subset E of $\mathbf{R}$, then it is continuous on E in the ordinary sense.

Proof. Let (x_n) be any convergent sequence with $\lim_{k\to\infty} x_k = x_0$. Then the sequence $(x_1, x_0, x_2, x_0, ..., x_0, x_n, x_0, ...)$ also converges to x_0, hence $\lim_{k\to\infty} \Delta x_k = 0$. So the sequence $(x_1, x_0, x_2, x_0, ..., x_0, x_n, x_0, ...)$ is forward convergent to 0 hence, by the hypothesis, the sequence

$$(f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_0), f(x_n), f(x_0), ...)$$

is forward convergent to 0. It follows from this that the sequence $(f(x_n))$ converges to $f(x_0)$. This completes the proof of the theorem.

The converse is not always true for the function $f(x) = x^2$ is an example since the sequence $(\sqrt{n})$ is forward convergent to 0 while $(f(\sqrt{n})) = (n)$ is not forward convergent to 0. □

Now we state the following straightforward result related to statistical continuity.

Corollary 2 If f is forward continuous, then it is statistically continuous.

Although the following result seems to be obvious, we state it to include a more general case.

Corollary 3 If f is forward continuous, then it is G-continuous for any regular subsequential method G.

Theorem 4 Forward continuous image of any forward compact subset of $\mathbf{R}$ is forward compact.

Proof. Let f be a forward continuous function and E be a forward compact subset of $\mathbf{R}$. Take any sequence $\mathbf{y} = (y_n)$ of terms in $f(E)$. Write $y_n = f(x_n)$ where $x_n \in E$ for each $n \in \mathbf{N}$. Forward compactness of E implies that there is a subsequence $\mathbf{z} = (z_k) = (x_{n_k})$ of $\mathbf{x}$ with $\lim_{k\to\infty} \Delta z_k = 0$. Since f is forward continuous, $(t_k) = f(\mathbf{z}) = (f(z_k))$ is forward convergent to 0. Thus (t_k) is a subsequence of the sequence $f(\mathbf{x})$ with $\lim_{k\to\infty} \Delta t_k = 0$. This completes the proof of the theorem. □

Corollary 5 Forward continuous image of any compact subset of $\mathbf{R}$ is compact.

The proof of this theorem follows from the preceeding theorem.

Theorem 6 If a function f on a subset E of $\mathbf{R}$ is uniformly continuous, then it is forward continuous on E.

Proof. Let f be a uniformly continuous function and $\mathbf{x} = (x_n)$ be any sequence of points in E such that $\lim_{n\to\infty} \Delta x_n = 0$. To prove that $(f(x_n))$ is forward convergent to 0, take any $\varepsilon > 0$. Uniform continuity of f implies that there exists a $\delta > 0$, depending on ε, such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Since (x_n) is forward convergent to 0, for this $\delta > 0$, there exist an $N = N(\delta) = N_1(\varepsilon)$ such that $|\Delta x_n| < \delta$ whenever $n > N$. Hence $|\Delta f(x_n)| < \varepsilon$ if $n > N$. It follows from this that $(f(x_n))$ is forward convergent to 0. This completes the proof of the theorem. □

It is well known that any continuous function on a compact subset E of $\mathbf{R}$ is also uniformly continuous on E. It is also true for a regular subsequential method G that any forward continuous function on a G-sequentially compact subset E of

R is also uniformly continuous on E (see [6]). Furthermore, for forward continuous functions, we have the following:
Theorem 7 If a function is forward continuous on a forward compact subset E of **R**, then it is uniformly continuous on E.

Proof. Suppose that f is not uniformly continuous on E so that there exist an $\epsilon_0 > 0$ and sequences (x_n) and (y_n) of points in E such that

$$|x_n - y_n| < 1/n$$

and

$$|f(x_n) - f(y_n)| \geq \epsilon_0$$

for all $n \in \mathbf{N}$. Since E is forward compact, there is a subsequence of (x_{n_k}) of (x_n) that is forward convergent to 0. Since E is forward compact, there is a subsequence of $(y_{n_{k_j}})$ of (y_{n_k}) that is forward convergent to 0. It is clear that the corresponding sequence $(x_{n_{k_j}})$ is also forward convergent to 0, since $(y_{n_{k_j}})$ is forward convergent to 0 and

$$|x_{n_{k_j}} - x_{n_{k_{j+1}}}| \leq |x_{n_{k_j}} - y_{n_{k_j}}| + |y_{n_{k_j}} - y_{n_{k_{j+1}}}| + |y_{n_{k_{j+1}}} - x_{n_{k_{j+1}}}|.$$

Now define a sequence $\mathbf{z} = (z_j)$ by setting $z_1 = x_{n_{k_1}}, z_2 = y_{n_{k_1}}, z_3 = x_{n_{k_2}}, z_4 = y_{n_{k_2}}, z_5 = x_{n_{k_3}}, z_6 = y_{n_{k_3}}$, and so on. Thus the sequence $\mathbf{z} = (z_j)$ defined in this way is forward convergent to 0 while $f(\mathbf{z}) = (f(z_j))$ is not forward convergent to 0. Hence this establishes a contradiction so this completes the proof of the theorem. □

It is a known result that uniform limit of a sequence of continuous functions is continuous. This is also true in case forward continuity, i.e. uniform limit of a sequence of forward continuous functions is forward continuous.
Theorem 8 If (f_n) is a sequence of forward continuous functions defined on a subset E of **R** and (f_n) is uniformly convergent to a function f, then f is forward continuous on E.

Proof. Let $\mathbf{x} = (x_n)$ be any sequence of points in E such that $\lim_{n\to\infty} \Delta x_n = 0$. Let $\varepsilon > 0$. Then there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in E$ whenever $n \geq N$. As f_N is forward continuous, there exists a positive integer N_1, depending on ε, and greater than N such that $|f_N(x_{n+1}) - f_N(x_n)| < \frac{\varepsilon}{3}$ for $n \geq N_1$. Now for $n \geq N_1$ we have

$$|f(x_{n+1}) - f(x_n)| \leq |f(x_{n+1}) - f_N(x_{n+1})| + |f_N(x_{n+1}) - f_N(x_n)| + |f_N(x_n) - f(x_n)|$$
$$\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This completes the proof of the theorem. □

References

[1] J.Antoni, On the A-continuity of real functions II, Math. Slovaca, **36**, No.3, (1986), 283-287. MR **88a**:26001

[2] J.Antoni and T.Salat, On the A-continuity of real functions, Acta Math. Univ. Comenian. **39**, (1980), 159-164. MR **82h**:26004

[3] J.Borsik and T.Salat, On F-continuity of real functions, Tatra Mt. Math. Publ. **2**, 1993, 37-42. MR **94m**:26006

[4] R.C.Buck, Solution of problem 4216 Amer. Math. Monthly **55**, (1948), 36. MR **15**: 26874

[5] J.Connor, Grosse-Erdmann K.-G. Sequential definitions of continuity for real functions, Rocky Mountain J. Math., **33**, (1), (2003), 93-121. MR **2004e**:26004

[6] H.Çakallı, Sequential definitions of compactness, Applied Mathematics Letters, 21 , No:6, 594-598, (2008)

[7] ...., Slowly oscillating continuity, Abstract and Applied Analysis, Volume 2008, Article ID 485706.

[8] H.Fast, Sur la convergence statistique, Colloq. Math. **2**, (1951), 241-244. MR **14**:29c

[9] J.A.Fridy, On statistical convergence, Analysis, **5**, (1985), 301-313 . MR **87b**:40001

[10] T.B.Iwinski, Some remarks on Toeplitz methods and continuity, Comment.Math. Prace Mat. **17**, 1972, 37-43. MR **48**759

[11] E.Öztürk, On almost-continuity and almost A-continuity of real functions, Comm.Fac.Sci. Univ.Ankara Ser. A1 Math. **32**, 1983, 25-30. MR **86h**:26003

[12] E.C.Posner, Summability preserving functions, Proc.Amer.Math.Soc. **12**, 1961, 73-76. MR **22**12327

[13] E. Savaş and G. Das, On the A-continuity of real functions, İstanbul Univ. Fen Fak. Mat Derg. **53**, (1994), 61-66. MR **97m**:26004

[14] I.J.Schoenberg, The integrability of certain functions and related summability methods, Amer. Math. Monthly **66**, 1959, 361-375. MR **21**:3696

[15] E. Spigel and N. Krupnik, On the A-continuity of real functions, J.Anal. **2**, (1994), 145-155. MR **95h**:26004

[16] V.K.Srinivasan, An equivalent condition for the continuity of a function, Texas J. Sci. **32**, 1980, 176-177. MR **81f**:26001

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