On the solvability of an inverse fractional abstract Cauchy problem

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Abstract

This note is devolved to study an inverse Cauchy problem in a Hilbert space H for fractional abstract differential equations of the form;

$$\frac{d^\alpha u(t)}{dt^\alpha} = A\ u(t) + f(t)\ g(t),$$

with the initial condition $u(0) = u_0 \in H$ and the overdetermination condition:

$$(u(t), v) = w(t),$$

where $(.,.)$ is the inner product in H, f is a real unknown function w is a given real function, u_0, v are given elements in H, g is a given abstract function with values in H, $0 < \alpha \leq 1$, u is unknown, and A is a linear closed operator defined on a dense subset of H.

It is supposed that A generates a semigroup. An application is given to study an inverse problem in a suitable Sobolev space for general fractional parabolic partial differential equations with unknown source functions.

Keywords: Inverse fractional Cauchy problem, solutions in Hilbert space, fractional parabolic partial differential equations.
Mathematics Subject Subject Classifications: 45D05, 47D09, 34 G10, 49 K 22, 35 K 90.

1. Introduction

Successful utilization of any fractional differential equation as a modeling tool requires results about existence, uniqueness and regularity properties of the solution under sufficiently general assumptions.
The general form of the equation is known and the details must be determined by reconciling the model with the observation of the process. In other words an inverse problem must be solved to find, on the basis of the observation, the coefficients, free term, the right-hand side, and sometimes, initial and boundary conditions.
Several authors [1-5] studied the unique solvability of inverse problems for various parabolic equations with unknown source functions under an integral overdetermination condition.
Cannon and Duchateu considered the identification of an unknown state - dependent source term in the heat equation [6].
In this note, the following general model is considered:

$$\frac{d^\alpha u(t)}{dt^\alpha} = A\ u(t) + f(t)g(t), \tag{1.1}$$

$$u(0) \ = \ u_0 \tag{1.2}$$

where u_0 is a given element in a real Hilbert space H, g is a given abstract function defined on an interval $J = [0, T], (T > 0)$ with values in H, f is an unknown real function, $0 < \alpha \leq 1$ and A is a linear closed operator defined on a dense subset $D(A)$ in H into H:
It is assumed that A generates an analytic semigroup $Q(t)$. This condition implies $\|Q(t)\| \leq \gamma$ for all $t \geq 0$, where $\|.\|^2 = (.,.)$, $(.,.)$ is the inner product in H and γ is a positive constant.

In section 2, the inverse Cauchy problem is studied under the overdetermination condition:

$$(u(t), v) \; = \; w(t) \quad , \tag{1.3}$$

where v is a given element in H and w is a given real function defined on J.

We shall suppose that the adjoint operator A^* of the closed operator A exists and that if

$$\frac{d^\alpha \phi(t)}{dt^\alpha} \; = \; \psi(t) \; ,$$

then

$$\phi(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \psi(s) ds,$$

where $\Gamma(\alpha)$ is the gamma function, $0 < \alpha \leq 1, \phi, \psi$ are abstract functions of t with values in H and the integral is taken in Bochner's sense [7].

In section 3 an application is given to the inverse Cauchy problem for equations of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} \; + \sum_{|q| \leq 2m} a_q(x) D^q u(x,t) = f(t) g(x,t) \; , \tag{1.4}$$

with the initial condition

$$u(x, 0) \; = \; u_0(x) \quad , \tag{1.5}$$

and the integral overdetermination condition

$$\int_Q u(x,t) v(x) dx \; = \; w(t), \tag{1.6}$$

where $q = (q_1, ..., q_n)$ is an n-dimensional multi-index, $x \in G \subset R^n$, R^n is the n-dimensional Euclidean space, G is a bounded region with smooth boundary $\partial G, D^q = D_1^{q_1} ... D_n^{q_n}$,

$$D_j = \frac{\partial}{\partial x_j}, \quad j \; = \; 1, ..., n, \quad |q| = q_1 + ... \; q_n$$

and the considered equation is fractional uniformly parabolic. In other words

$$(-1)^m \sum_{|q|=2m} a_q(x) y^q \geq \; c|y|^{2m},$$

for all $x \in \overline{G} = GU\partial G, y \in R^n$, where $|y|^2 = y_1^2 + ... + y_n^2$, $y^q = y_1^q ... y_n^{q_n}$ and c is a positive constant.
It is assume that $a_q \in C^{2m}(G)$, for all $|q| \leq 2m$, where $C^j(G)$ is the set of all continuous real-valued functions defined on G, which have continuous partial derivatives of order less than or equal to j.

The functions u_0, v and w are given. The unknown functions u and f are determined in a suitable space.

2.Abstract inverse problem

A pair of functions $\{u, f\}$ is said to be a strictly solution of the inverse problem (1.1)-(1.3) if

$$u \in D(A), \frac{d^\alpha u(t)}{dt^\alpha} \in H$$

for each $t \in (0, T], f \in C(J)$ and the relations (1.1)-(1.3) are satisfied. In this case we say that the inverse problem (1.1)-(1.3) is solvable.
We shall assume the following conditions;
A_1**:** $u_0, v \in D(A), g(t) \in D(A)$ for all $t \in J$,
A_2**:** $|g_1(t)| \geq g_0, t \in J$, where $g_1(t) = (g(t), v)$ and g_0 is a positive constant,
A_3**:** The abstract functions g and Ag are continuous on J with respect to the norm in H,
A_4**:** $\frac{d^\alpha w}{dt^\alpha} \in C(J)$.
Let us consider the following equation:

$$f \ = \ h + Pf \ , \tag{2.1}$$

where

$$h(t) = \frac{1}{g_1(t)} \frac{d^\alpha w(t)}{dt^\alpha}$$

and P is a linear operator defined on $C(J)$ with values:

$$(Pf)(t) = -\frac{1}{g_1(t)} \quad (Au(t), v) \tag{2.2}$$

We shall prove now the equivalence between the inverse problem (1.1)-(1.3) and (2.1).

Theorem 2.1. Suppose that the conditions $(A_1 - A_4)$ are satisfied. Then the following assertions are valid :

(I) If the inverse problem (1.1) is solvable, then so equation (2.1) has a solution $f \in C(J)$,

(II) If equation (2.1) has a solution $f \in C(J)$ and the compatibility condition

$$(u_o, v) = w(0), \tag{2.3}$$

holds, then the inverse problem (1.1) - (1.3) is solvable.

Proof. Assume that the inverse problem (1.1) - (1.3) is solvable. Multiplying both sides of (1.1) by v scalarly in H, we obtain the relation

$$\frac{d^\alpha}{dt^\alpha}(u(t), v) = (Au(t), v) + f(t)g_1(t). \tag{2.4}$$

From (2.2) and (2.4), one gets

$$f \;=\; Pf + \frac{1}{g_1}\,\frac{d^\alpha w}{dt^\alpha}$$

This means that f solves equation (2.1).

To prove assertion (II), we notice that by the assumption, equation (2.1) has a solution $f \in C(J)$. When inserting this function in (1.1), the resulting problem (1.1), (1.2) can be treated as a direct problem having a unique solution u. Using previous results [8], this solution is given by

$$u(t) = \int_0^\infty \zeta_\alpha(\theta) Q(t^\alpha \theta) u_0 \, d\theta$$

$$+\,\alpha \int_0^t \int_0^\infty \theta (t-s)^{\alpha-1} \zeta_\alpha(\theta) Q((t-s)^\alpha \theta) f(s) g(s) d\theta \, ds \tag{2.5}$$

let us prove now that u satisfies the overdetermination condition (1.3). In this case u and f are known, consequently (2.4) will represent an identity,

$$f(t)\; g_1(t) = \frac{d^\alpha w(t)}{dt^\alpha} - (Au, v). \tag{2.6}$$

Subtracting equation (2.4) from (2.6), one gets

$$\frac{d^\alpha w(t)}{dt^\alpha} \;=\; \frac{d^\alpha}{dt^\alpha}\;(u(t), v)).$$

applying the fractional integral of order α and taking into account the compatibility condition (2.3), we find out that u satisfies the overdetermination condition (1.3) and that the pair $\{u, f\}$ is a strictly solution of the inverse problem (1.1) - (1.3). This completes the proof of the theorem.

Theorem 2.2. Let the conditions $(A_1 - A_4)$ and the compatibility condition (2.3) hold, then there exists a unique strictly solution of the inverse problem (1.1) - (1.3).

Proof. Using (2.1), (2.2) and (2.5), one obtains (formally) the following Volterra integral equation

$$f(t) = \psi(t) - \int_0^t (t-s)^{\alpha-1} K(t,s) f(s) ds, \tag{2.7}$$

where

$$\psi(t) = h(t) - \frac{1}{g_1(t)} \int_0^\infty (Q(t^\alpha\theta)u_0, A^*v) d\theta,$$

$$K(t,s) = \frac{\alpha}{g_1(t)} \int_0^\infty \theta\zeta_\alpha(\theta)(Q(t-s)^\alpha\theta)\ g(s), A^*v)\ d\theta.$$

According to conditions A_2, A_3 and A_4, the functions $g_1^{-1}(t)$ and h(t) are continuous on J. We shall prove now that the function ψ is continuous on J. In fact;

$$|\int_0^\infty |\zeta_\alpha(\theta)(\int_{t_1}^{t_2} \frac{d}{dt} Q(t^\alpha\theta)u_0\ dt\ ,\ A^*v)\ d\theta|$$

$$= |\int_0^\infty\ \zeta_\alpha(\theta)(\int_{t_1}^{t_2} \alpha t^{\alpha-1}\theta Q(t^\alpha\theta)Au_0 dt, A^*v)d\theta|$$

$$\leq \gamma\|Au_0\|\ \|A^*v\|\ (t_2^\alpha - t_1^\alpha),$$

$$t_2 > t_1, t_1, t_2 \in J.$$

We shall prove that equation has a unique solution $f \in C(J)$.

Using the method of successive approximations, we set

$$f_{n+1}(t) = \psi(t) - \int_0^t (t-s)^{\alpha-1} K(t_1 s) f_n(s) ds,$$

$$f_0(t) = 0, \text{for all } t \in J, \quad n = 1, 2, ...$$

It is easy to see that

$$|f_2(t) - f_1(t)|\ \leq\ M\ t^\alpha,$$

where

$$M = \frac{\gamma}{g_0} \sup_t \|g(t)\| \, \|A^* v\|$$

By induction, one gets

$$|f_{n+1}(t) - f_n(t)| \leq \frac{M^n t^{n\alpha} (\Gamma(\alpha))^n}{\Gamma(n\alpha + 1)} \tag{2.8}$$

It can be proved that all the functions $f_{n+1}(t) - f_n(t)$ are continuous on J, (comp. [9-15]). Using (2.8), we see that the series $\sum_{k=1}^{\infty}[f_k(t) - f_{k-1}(t)]$ uniformly converges on J to a continuous function $f(t)$, which represents the unique solution of (2.7).

According to theorem (2.1) this confirms that the inverse problem (1.1)-(1.3) is solvable.

To prove the unique solution of (2.7). According to theorem (2.1) this confirms that the inverse problem (1.1)-(1.3) is solvable. to prove the uniqueness of this solution, we assume to the contrary that there were two different solutions $\{v_1, f_1\}$ and $\{u_2, f_2\}$ of the inverse problem (1.1) - (1.3). We claim that in this case $f_1 \neq f_2$ for all points of J. In fact if $f_1 = f_2$ on J then applying the uniqueness theorem to the corresponding direct problem (1.1), (1.2), we would have $u_1 = u_2$. Since both pairs satisfy identity (2.4), the functions f_1 and f_2 give two different solutions of equation (2.7). But this contradicts the uniqueness of solutions to the Volterra integral equation (2.7). This completes the proof of the theorem.

3. Inverse mixed problem

Let $W^m(G)$ be the completion of the space $C^m(G)$ with respect to the norm

$$\|v\|_m^2 = \sum_{|q| \leq m} \int_G |D^q v(x)|^2 dx. \tag{3.1}$$

Denote by $W_0^m(G)$ the completion of the space $C_0^m(G)$ with respect to the norm (3.1), (where $C_0^m(G)$) is the set of all functions $f \in C^m(G)$ with compact supports in $\overline{G}$.

Let $L^2(G)$ be the space of all square integrable functions on G.

The inverse problem (1.4)-(1.6) can be written in the abstract form (1.1)-(1.3), where A is the operator defined by $Au = u_1$,

$$u_1(x,t) = -\sum_{|q|\leq 2m} a_q(x) D^q u(x,t) \tag{3.2}$$

The domain of definition of A is given by

$$D(A) = W^{2m}(G) \ \cap W_0(G)$$

The considered set D(A) is dense in $L_2(G)$ and the closed operator A defined by (3.2) generates a semigroup [16], [17]. The adjoint operator A^* is given by $A^*u = u_2$, where

$$u_2(x,t) = -\sum_{|q|\leq 2m} (-1)^{|q|} D^q [a_q(x) u(x,t)].$$

Applying theorem (2.1) and (2.2) we can see that the inverse problem (1.4)-(1.6) is uniquely solvable.

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