CESÀRO SEMICONSERVATIVE FK SPACES

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Abstract. In this paper we call an FK space X containing ϕ Cesàro semiconservative space if $X^f \subset \sigma s$ holds. Therefore we give some characterizations of these spaces.

Key Words: FK spaces, Semiconservative FK space.
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1.Introduction.

Snyder and Wilansky give the definition of semiconservative FK space and investigate the properties of this space in [7] , [8] . In those papers, an FK space X containing ϕ is called semiconservative space if $X^f \subset cs$ holds. Here replacing cs by σs, we give a new definition called as Cesàro semiconservative FK space.

2. Notions and Definitions.

Let w denote the space of all real or complex-valued sequences. It can be topologized with the seminorms $p_i(x) = |x_i|$, $(i = 1, 2, ...)$, and any vector subspace of w is called a sequence space. A sequence space X, with a vector space topology τ is a K space provided that the inclusion mapping $I : (X, \tau) \to w, ; I(x) = x$ is continuous. If, in addition, τ is complete, metrizable and locally convex then (X, τ) is called an FK space. So an FK space is a complete, metrizable local convex topological vector space of sequences for which the coordinate functionals are continuous. An FK space whose topology is normable is called a BK space. The basic properties of such spaces can be found in [8] , [9] and [10] .

By m, c_0 we denote the spaces of all bounded sequences,null sequences, respectively. These are FK spaces under $\|x\| = \sup_n |x_n|$. By l and cs we shall denote the space of all absolutely summable sequences and convergent series, respectively. The sequences spaces

$$h = \left\{ x \in w : \lim_j x_j = 0, \text{ and } \sum_{j=1}^{\infty} j\,|\Delta x_j| < \infty \right\},$$

$$q = \left\{ x \in w : \sup_j |x_j| \langle \infty \text{ and } \sum_{j=1}^{\infty} j\,\left|\Delta^2 x_j\right| < \infty \right\},$$

$$\sigma b=\left\{x\in w:\sup_n\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\right|<\infty\right\},$$

$$\sigma s=\left\{x\in w:\lim_n\left|\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_j\right|\text{exists}\right\},$$

and

$$\sigma_0=\left\{x\in w:\lim_n\left|\frac{1}{n}\sum_{k=1}^{n}x_k\right|=0\right\}$$

are FK spaces with the norms

$$\|x\|_h=\sum_{j=1}^{\infty}j\left|\Delta x_j\right|+\sup_j\left|x_j\right|,$$

$$\|x\|_q=\sum_{j=1}^{\infty}j\left|\Delta^2 x_j\right|+\sup_j\left|x_j\right|,$$

$$\|x\|_{\sigma b}=\sup_n\left|\frac{1}{n}\sum_{j=1}^{n}\sum_{j=1}^{k}x_j\right|,$$

and

$$\|x\|_{\sigma_0}=\sup_n\frac{1}{n}\left|\sum_{k=1}^{n}x_k\right|$$

respectively, where $\Delta x_j=x_j-x_{j+1},\Delta^2 x_j=\Delta x_j-\Delta x_{j+1}$. The space $q\cap c_0$ is denoted by q_0. Under the norm $\|.\|_q$, q_0 is a BK space, (see [1], [2]).

Throughout the paper e denotes the sequence of ones, $(1,1,...,1,...)$; δ^j, $(j=1,2,...)$, the sequence $(0,0,...,0,1,0,...)$ with the one in the $j-$th position. Let $\phi:=l.hull\left\{\delta^k:k\in N\right\}$ and $\phi_1=\phi\cup\{e\}$. The topological dual of X is denoted by X'. The space X is said to have AD if ϕ is dense in X and an FK space X is said to have AK or be an AK space, if $X\supset\phi$ and for each $x\in X$, $x^{(n)}\to x$, $(n\to\infty)$, in X, where $x^{(n)}=\sum_{k=1}^{n}x_k\delta^k=(x_1,x_2,...,x_n,0,...)$. In addition an FK space is said to have σK space if $X\supset\phi$ and for each $x\in X$, $\frac{1}{n}\sum_{k=1}^{n}x^{(k)}\to x,(n\to\infty)$. Every AK space is a σK space. For example w, h, c_0 σ_0 are AK spaces while q_0, σs are σK spaces ([1], [2], [8]). In addition, every σK space is an AD space.

Let X be an FK space containing ϕ. Then

$$X^f=\left\{\left\{f\left(\delta^k\right)\right\}:f\in X'\right\}.$$

In addition

$$X^{\beta}=\left\{x:\sum_{k=1}^{\infty}x_ky_k \text{ exists for every } y\in X\right\},$$

$$X^{\sigma}=\left\{x:\lim_n\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_jy_j \text{ exists for every } y\in X\right\},$$

$$X^{\sigma b}=\left\{x:\sup_n\frac{1}{n}\left|\sum_{k=1}^{n}\sum_{j=1}^{k}x_jy_j\right|<\infty \text{ for every } y\in X\right\}.$$

Let $E,\ E_1$ be sets of sequences. Then for $k=f,\ \beta,\ \sigma,\ \sigma_b$

$$(a)\ E\subset E^{kk},\quad (b)\ E^{kkk}=E^{k},\quad (c)\text{ if } E\subset E_1 \text{ then } E_1^{k}\subset E^{k}$$

holds.

Theorem 2.1. *Let X be an FK space containing ϕ. Then*

(i) $X^{\beta}\subset X^{\sigma}\subset X^{\sigma b}\subset X^{f}$,

(ii) If X is σK space then $X^{f}=X^{\sigma}$,

(iii) If X is an AD space then $X^{\sigma}=X^{\sigma b}$.

Proof.(ii) Let $u\in X^{\sigma}$ and define $f(x)=\lim_n\frac{1}{n}\sum\limits_{k=1}^{n}\sum\limits_{j=1}^{k}x_ju_j$ for $x\in X$. Then $f\in X'$; by the Banach-Steinhause Theorem ([8] ,1.0.4).

Also $f(\delta^{p})=\lim_n\frac{1}{n}\left(n-(p-1)\right)u_p=u_p\ (p<n)$ so $u\in X^{f}$. Thus $X^{\sigma}\subset X^{f}$.

Now we show that $X^{f}\subset X^{\sigma}$. Let $u\in X^{f}$. Since X is σK space

$$f(x)=\lim_n\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_jf(\delta^{j})=\lim_n\frac{1}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}x_ju_j$$

for $x\in X$, then $u\in X^{\sigma}$. Hence $X^{f}=X^{\sigma}$.

(iii) Let $u\in X^{\sigma b}$ and define $f_n(x)=\frac{1}{n}\sum\limits_{k=1}^{n}\sum\limits_{j=1}^{k}x_ju_j$ for $x\in X$. Then $\{f_n\}$ is pointwise bounded, hence equicontinuous by ([8] , 7.0.2) .

Since $\lim_n f_n(\delta^{p})=u_p\ (p<n)$ then $\phi\subset\{x:\lim_n f_n(x) \text{ exists}\}$. Hence $\{x:\lim_n f_n(x) \text{ exists}\}$ is closed subspace of X by the Convergence Lemma, ([8] , 1.0.5,7.0.3). Since X is an AD space then $X=\{x:\lim_n f_n(x) \text{ exists}\}=\overline{\phi}$ and then $\lim_n f_n(x)$ exists for all $x\in X$. Thus $u\in X^{\sigma}$. The opposite inclusion is trivial.

(i) $\overline{\phi}\subset X$ by the hypothesis. Since $\overline{\phi}$ is σK space, then $X^{\sigma b}\subset\left(\overline{\phi}\right)^{\sigma b}=\left(\overline{\phi}\right)^{\sigma}=\left(\overline{\phi}\right)^{f}=X^{f}$ by $(ii),(iii)$ and ([8] , 7.2.4) .

Let $A = (a_{ij})$ be an infinite matrix. The matrix A may be considered as a linear transformation of sequences (x_k) by the formula $y = Ax$, where $y_i = \sum_{j=1}^{\infty} a_{ij}x_j$,$(i = 1, 2, ...)$.

For an FK space (E, u) we consider the summability domain

$E_A := \{x \in w : Ax \in E\}$. Then E_A is an FK space under the seminorms $p_i = |x_i|, (i = 1, 2, ...)$, $h_i(x) = \sup_m \left|\sum_{j=1}^{m} a_{ij}x_j\right|, (i = 1, 2, ...)$ and $(u \circ A)(x) = u(Ax)$, [8].

Recall that, given a matrix A with $l_A \supset \phi$ is called $l-replaceable$ if there is a matrix $B = (b_{nk})$ with $l_B = l_A$ and $\sum_{n=1}^{\infty} b_{nk} = 1$, for all $k \in N$, [6].

An FK space containing ϕ_1 is called Cesàro conull if

$f(e) = \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} f(\delta^j)$,for all , $f \in X'$, [5].

In addition an FK space X is called semiconservative if $X^f \subset cs$, this means that $X \supset \phi$ and $\sum_{j=1}^{\infty} f(\delta^j)$ is convergent for each $f \in X'$, [7].

3.Cesàro semiconservative FK Spaces

In this section we extend the notation of the semiconservative FK space introduced by Snyder and Wilansky [7] to the concept of Cesàro semiconservative FK Space and we investigate the properties of these spaces.

Definition 3.1. *An FK space X is called Cesàro semiconservative if $X^f \subset \sigma s$, where $X^f \subset \sigma s$ if and only if $e^{(k)}$ is weakly Cesàro Cauchy i.e. $\left\{\frac{1}{n} \sum_{k=1}^{n} f(e^{(k)})\right\}$ is convergent for each $f \in X'$ equivalently $\lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} f(\delta^j)$ exists.*

For example c_0, σ_0 are Cesàro semiconservative FK spaces. Every semiconservative FK space is an Cesàro semiconservative FK space.But the following example shows that every Cesàro semiconservative FK space is not a semiconservative space. Before this example we shall give some theorems.

Theorem 3.2. *If a matrix A is $l-replaceable$ then l_A is not Cesàro semiconservative FK space.*

Proof. If A does $l-$replaceable then there is $f \in l_A'$ such that $f(\delta^j) = 1$ for all $j \in N$, [6].

Hence $\lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} f(\delta^j)$ does not exist since $\frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} f(\delta^j) = \frac{n+1}{2}$, so l_A is not Cesàro semiconservative space.

Theorem 3.3. *If X_A is Cesàro conull FK space then it is Cesàro semiconservative space.*

Proof. Suppose that X_A is Cesàro conull. Then

$$f(e) = \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} f\left(\delta^j\right),$$

for all $f \in X'_A$. Hence $X_A^f \subset \sigma s$.

Now we present the examples promised in this section.

Example 3.4. *Define the sequence Ax by $(Ax)_j = x_j - x_{j-1}$ $(x_0 = 0)$ if j is square, and 0 otherwise. Then l_A is Cesàro semiconservative space but not semiconservative space.*

Proof. l_A is Cesàro conull FK space by ([5] ,Example 3.2), so l_A is Cesàro semiconservative space by Theorem 3.3.

Now we show that l_A is not semiconservative space.To see this let $B := A^T$. Then $\sum_{k=1}^{\infty} \left| \sum_{i=n}^{\infty} b_{ik} \right| = 1$ if n is sequare ,otherwise 0. Thus $\lim_n \sum_{k=1}^{\infty} \left| \sum_{i=n}^{\infty} b_{ik} \right|$ does not exist and $B \notin (l^\infty; cs) = \left(l^\beta : cs\right)$ by ([8] , 8.5.8) and then l_A is not semiconservative space by ([8] ,9.4.4).

Theorem 3.5. (i) *A closed subspace, containing ϕ, of a Cesàro semiconservative space is a Cesàro semiconservative space.*

(ii) *An FK space that contains a Cesàro semiconservative space must be a Cesàro semiconservative space.*

(iii) *A countable intersection of Cesàro semiconservative spaces is a Cesàro semiconservative space.*

Proof. (i) is true since $Y^f = X^f$ ([8] , Theorem 7.2.6). (ii) holds since $Y^f \subset X^f \subset \sigma s$. To prove (iii); First the intersection $X = \bigcap X_n$ is an FK space by ([8] , Theorem 4.2.15). Every $f \in X'$ can be written $f = \sum_{k=1}^{m} g_k$ where each $g_k \in X'_n$ for some n by ([8] , 4.0.3, 4.0.8).

Theorem 3.6 z^σ is a *Cesàro semiconservative space if and only if* $z \in \sigma s$.

Proof. Let z^σ be a Cesàro semiconservative space. Then $z^{\sigma f} \subset \sigma s$. Since z^σ is a σK space by [5], we have $z^{\sigma f} = z^{\sigma\sigma}$. So since $\{z\} \subset z^{\sigma\sigma} \subset \sigma s$, we get $z \in \sigma s$.

Now let $z \in \sigma s$. Then $q = \sigma s^\sigma \subset z^\sigma$ [1] and hence $z^{\sigma\sigma} \subset q^\sigma = \sigma s$. Since z^σ is a σK space, then $z^{\sigma f} = z^{\sigma\sigma} \subset \sigma s$.

Example 3.7. *σs is not Cesàro semiconservative space. Because $\sigma s = e^\sigma$ and $e \notin \sigma s$.*

Theorem 3.8. *(i) Every Cesàro semiconservative space contains q_0.*

(ii) The intersection of all Cesàro semiconservative spaces is q_0.

(iii) q_0 is not Cesàro semiconservative space.

(iv) There is no smallest Cesàro semiconservative space.

Proof. (i) Let X be a Cesàro semiconservative space. Then $X^f \subset \sigma s \subset \sigma b = q_0^\sigma$, [1] and since q_0 is a σK space then $X^f \subset q_0^\sigma = q_0^f$. So,since q_0 is an AD space, we obtain $q_0 \subset X$ by ([8], Theorem 8.6.1).

(ii) Let the intersection of all Cesàro semiconservative spaces I. We get $I \subset \cap \{z^\sigma : z \in \sigma s\} = \sigma s^\sigma = q$ using Theorem 3.6. Also $I \subset c_0$ since c_0 is Cesàro semiconservative space so $I \subset q \cap c_0 = q_0$. The opposite inclusion is by (i).

(iii) Since $q_0^f = q_0^\sigma = \sigma b \not\subseteq \sigma s$ then q_0 is not Cesàro semiconservative space.

(iv) By (ii) and (iii).

Example 3.9. *q and σb are not Cesàro semiconservative spaces.*

Proof. q_0 and σs are closed subspaces of, respectively, q and σb.Then since q_0 and σs are not Cesàro semiconservative spaces. q and σb are not Cesàro semiconservative spaces by Theorem 3.5 (i).

$X^\sigma \subset \sigma s$ is not sufficent for X to be Cesàro semiconservative space since $q^\sigma = \sigma s$. This is not surprising since this condition holds for every space containing e.

Definition 3.10. *An FK space is called bounded convex Cesàro semiconservative if it is Cesàro semiconservative space and includes q.*

Since q_0 is an AD space then $X \supset q_0$ if and only if $X^f \subset \sigma b$ by ([8], 8.6.1). Thus $X \supset q$ if and only if $X^f \subset \sigma b$ and $e \in X$, by ([8], 8.3.7). However X is bounded convex Cesàro semiconservative space if and only if $X^f \subset \sigma s$ and $e \in X$, also if and only if X is Cesàro semiconservative space and $e \in X$.

The definition of Cesàro conull FK space X which $X \supset \phi$, can be given as follows by using Cesàro semiconservative; A Cesàro semiconservative space X is called Cesàro conull, if $f(e) = \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} f(\delta^j)$, for all $f \in X'$. A Cesàro semiconservative space need not contain e but must contain e, if it is Cesàro conull. A Cesàro conull space is automatically bounded convex Cesàro semiconservative space.

4. A Relationship Between the Distinguished Subsets and Cesàro Semiconservative FK spaces.

In this section we give the relation between the distinguished subspaces which are σF^+, σF, σB^+, σB and Cesàro semiconservative and bounded convex Cesàro semiconservative FK spaces. Before this we shall give the definition of the distinguished subsets σF^+, σF, σB^+, σB.

Let X be an FK space containing ϕ. Then we define

$$\sigma F^{+}(X) = \left\{ x : \lim_n \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{k} x_j f\left(\delta^j\right) \text{ exists for all } f \in X' \right\}$$
$$= \left\{ x : \{x_n f(\delta^n)\} \in \sigma s \text{ for all } f \in X' \right\}$$

$$\sigma B^{+}(X) = \left\{ x : \left\{ \frac{1}{n} \sum_{k=1}^{n} x^{(k)} \right\} \text{is bounded in } X \right\}$$
$$= \left\{ x : \ \{x_n f(\delta^n)\} \in \sigma b \text{ for all } f \in X' \right\}.$$

Also $\sigma F = \sigma F^{+} \cap X$ and $\sigma B = \sigma B^{+} \cap X$, [4].

Theorem 4.1.*Let X be an FK space containing ϕ and $z \in w$. Then $z \in \sigma F^{+}$ if and only if $z^{-1}.X = \{x : z.x \in X\}$ is Cesàro semiconservative FK space, where, $z.x = \{z_n x_n\}$ in particular $e \in \sigma F^{+}$ if and only if X is Cesàro semiconservative FK space.*

Proof. Let $f \in (z^{-1}.X)'$. Then $f(x) = \alpha x + g(z.x)$, $\alpha \in \phi$, $g \in Y'$, by ([8], 4.4.10) and $f(\delta^n) = \alpha_n + g(z.\delta^n) = \alpha_n + g(z_n.\delta_n) = \alpha_n + z_n g(\delta^n)$. Hence, since $\alpha \in \phi \subset \sigma s$ then $\{f(\delta^n)\} \in \sigma s$ if and only if $\{z_n g(\delta^n)\} \in \sigma s$, i.e. $z \in \sigma F^{+}$.

Theorem 4.2.*Let X be an FK space containing ϕ and $z \in w$. Then $z \in \sigma F$ if and only if $z^{-1}.X$ is bounded convex Cesàro semiconservative FK space in particular $e \in \sigma F$ if and only if X is bounded convex Cesàro semiconservative.*

Proof. Let $z \in \sigma F$. Then $z \in X$ so $e \in z^{-1}.X$ and since $z \in \sigma F^{+}$, $z^{-1}.X$ is Cesàro semiconservative FK space by Theorem 4.1. Thus $z^{-1}.X$ is bounded convex Cesàro semiconservative FK space .

Let $z^{-1}.X$ is bounded convex Cesàro semiconservative FK space. Then $z^{-1}.X$ is Cesàro semiconservative and $e \in z^{-1}.X$ so $z \in X$. Thus since $z \in \sigma F^{+}$ by Theorem 4.1. and $z \in X$ then $z \in \sigma F$.

Theorems 4.3 and 4.4 have already been obtained by Buntinas [1] but we present here alternate proofs of them.

Theorem 4.3. *Let X be an FK space containing ϕ and $z \in w$. Then $z \in \sigma B^{+}$ if and only if $z^{-1}.X \supset q_0$, in particular $e \in \sigma B^{+}$ if and only if $X \supset q_0$.*

Proof. Let $f \in (z^{-1}.X)'$. Then $f(\delta^n) = \alpha_n + z_n g(\delta^n)$ by ([8] , 4.4.10) .Thus, since $\alpha \in \phi \subset \sigma s$ then $z \in \sigma B^{+}$ if and only if $\{z_n g(\delta^n)\} \in \sigma b$, i.e. $z \in \sigma B^{+}$.

Theorem 4.4. *Let X be an FK space containing ϕ and $z \in w$. Then $z \in \sigma B$ if and only if $z^{-1}.X \supset q$, in particular $e \in \sigma B$ if and only if $X \supset q_0$.*

Proof. Let $z \in \sigma B$. Then $z \in X$ so $e \in z^{-1}.X$ and $z \in \sigma B^{+}$. Thus $z^{-1}.X \supset q$ by Theorem 4.3.

Let $z^{-1}.X \supset q$ then $z^{-1}.X \supset q_0$ and $e \in z^{-1}.X$. Thus, since $z \in \sigma B^{+}$ by Theorem 4.3 and $z \in X$ then $z \in \sigma B$.

Theorem 4.5. *Let X be an FK space containing ϕ. Then X is Cesàro semiconservative space if and only if $\sigma F^{+} \supset q$.*

Proof. Let X be a Cesàro semiconservative FK space. Then $e \in \sigma F^{+}$ by Theorem 4.1. Since $e \in \sigma F^{+} = X^{f\sigma}$ [4] then $X^{f} \subset X^{f\sigma\sigma} \subset \{e\}^{\sigma}$ and so $q = \{e\}^{\sigma\sigma} \subset X^{f\sigma} = \sigma F^{+}$.

Let $\sigma F^{+} \supset q$. Then $e \in \sigma F^{+}$ and so X is Cesàro semiconservative FK space by Theorem 4.1.

Theorem 4.6. *Let Y be a Cesàro semiconservative FK space and Z an AD space. Suppose that for an FK space X, $X \supset Y.Z$. Then $X \supset Z$, where $Y.Z = \{y.z : y \in Y \ , z \in Z\}$.*

Proof. Let $z \in Z$. Then, since $X \supset Y.Z$, $z^{-1}.X \supset Y$. Thus, since Y is Cesàro semiconservative space then $z^{-1}.X$ is Cesàro semiconservative space by Theorem 3.5.(ii) and so $z \in \sigma F^{+}$ by Theorem 4.1. Hence $Z \subset \sigma F^{+} = X^{f\sigma}$ [4]. Thus $X^{f} \subset X^{f\sigma\sigma} \subset Z^{\sigma} \subset Z^{f}$ and so $Z \subset X$ by ([8], 8.6.1).

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