STAT 305/405 PRACTICE QUESTIONS FOR FINAL EXAM

1. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the **Gamma distribution** with unknown parameters α and β.

   [a] Find the first and second moments μ_1' and μ_2'.

   [b] Find the **method of moments estimators** of α and β in terms of the first and second sample moments m_1' and m_2'.

2. Do Exercise 10.67 on page 527 of our textbook.

3. Do Exercise 10.69 on page 527 of our textbook.

4. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample of size n from the **Poisson distribution** with unknown parameters λ.

   [a] Show that $S_n = Y_1 + \cdots Y_n$ has a Poisson distribution with parameter $n\lambda$.

   [b] State the Neyman-Pearson lemma.

   [c] Consider the hypotheses $H_0 : \lambda = 2$ versus $H_a : \lambda = 5$. Use the Neyman-Pearson lemma to show that the most powerful test with significance level α has a rejection region of the form $S_n > c$.

   [d] If $n = 4$ and $\alpha = 0.05$, find this rejection region. (Use Table 3 of Appendix 3).

   [e] Consider the hypotheses $H_0 : \lambda = \lambda_0$ versus $H_a : \lambda > \lambda_0$. Show that the Neyman-Pearson lemma gives us the uniformly most powerful test.

5. Let $Y_1, Y_2, \ldots, Y_n$ be a random sample from the normal distribution with known mean $\mu = 0$ and unknown variance σ^2.

   [a] Show that the maximum likelihood estimator of σ^2 is $\widehat{\sigma^2} = \frac{1}{n}\sum_{i=1}^{n} Y_i^2$

   [b] Show that, for each $i = 1, 2, \ldots, n$, the distribution of $\frac{Y_i^2}{\sigma^2}$ is χ_1^2 and deduce from this that $E(Y_i^2) = \sigma^2$

   [c] Prove that $\widehat{\sigma^2}$ is an unbiased estimator of σ^2.

   [d] Prove that $\widehat{\sigma^2}$ is a consistent estimator of σ^2. (Hint: Find the variance of $\frac{Y_i^2}{\sigma^2}$)

6. A coin is tossed independently 10 times to test the null hypothesis that the probability of a head is 0.5 versus the alternative hypothesis that the probability of a head is less that 0.5. The test statistic is Y, the number of heads out of 10 tosses. The rejection region is {0,1}; that is, reject the null if the number of heads observed is either 0 or 1.

   [a] Find the level of significance α of this test.

   [b] If the probability of a head is 0.1, find the power of the test.

7. Circle True or False. If your answer is correct, no explanation is necessary. If wrong, a good explanation may give you partial credit.

T **F** The probability that the null hypothesis is falsely rejected is equal to the power of the test.

T **F** A type I error occurs when the test statistics falls in the rejection region and the alternative hypothesis is false.

T **F** At a fixed level of significance α, an increase in sample size generally results in a decrease in the probability of making a type II error.

T **F** A type II error occurs when the alternative hypothesis is composite.

T **F** If the power of the test is 0.95, then the probability of making a Type II error is 0.05.

T **F** If the test statistic falls in the rejection region then it is impossible you made a type II error.

T **F** In general, if the test statistics does not fall in the rejection region, then we may conclude that the null hypothesis is true.

T **F** All Method of Moments estimators are unbiased.

T **F** The probability of making a type II error β may be computed from the level α by using $\alpha + \beta = 1$.

8. Let $\{y_1 = 10, y_2 = 11, y_3 = 12\}$ be a random sample of size $n = 3$ from a normal population. Note that $\overline{y} = 11$ and $s^2 = 1$. Find the method-of-moments estimates of μ and σ^2.

9. A binomial experiment consists of n trials resulting in observations $y_1, y_2, \ldots, y_n$, where $y_i = 1$ if the trial was a success (with unknown probability p) and $y_i = 0$ otherwise. The random variable $Y = Y_1 + Y_2 + \cdots + Y_n$ is binomial with p unknown and n known.

   [a] Find the likelihood function $L(p)$.

   [b] Show that is a sufficient statistic for estimating p.

   [c] Find the maximum likelihood estimator (MLE) for p.

10. Let $Y_1, Y_2, \ldots, Y_9$ be a random sample of size $n = 9$ from the **normal distribution** with unknown mean μ and known variance $\sigma^2 = 1$. We wish to test

$$H_0 : \mu = 4 \quad \text{versus} \quad H_A : \mu > 4$$

using the rejection region

$$\overline{y} > k$$

   [a] Find the value of k that makes the level of significance α equal to $\alpha = 0.01$

   [b] What sample size is needed to insure that $\beta \leq 0.1$, when the true value is $\mu = 5$

11. Assume that the weight in ounces Y of a "10-pound" bag of sugar is $N(\mu, 5)$. We want to use the LRT to test the hypothesis $H_0 : \mu = 162$ against the hypothesis $H_A : \mu \neq 162$. Note 10 pounds = 160 ounces.

   [a] $\Omega_0 =$ ______

   [b] $\Omega =$ ______

   [c] Show that the Likelihood Ratio Statistic is $\lambda = \exp\left[-\frac{n}{10}(\overline{y} - 162)^2\right]$

   [d] Show that the RR is $\left\{\overline{y} : \frac{|\overline{y}-162|}{\sigma/\sqrt{n}} > z_{\alpha/2}\right\}$

12. Find the rejection region of the likelihood ratio test for testing the simple null hypothesis

$$H_0 : \mu = \mu_0$$

against the composite

$$H_A : \mu \neq \mu_0$$

on the basis of a random sample of size n from a normal population with the known variance σ^2.

13. The number of successes in n independent trials is to be used to test the null hypothesis that the parameter p of a binomial population is $\frac{1}{2}$ against the alternative that it does not equal $\frac{1}{2}$.

[a] Find an expression for the likelihood ratio statistic.

[b] Use the result of part [a] to show that the Rejection Region of the Likelihood Ratio Test (LRT) can be written

$$y \ln y + (n - y) \ln(n - y) > k$$

where y is the observed number of successes.

[c] Considering the graph of $f(y) = y \ln y + (n-y) \ln(n-y)$, in particular the minimum and symmetry, show that the Rejection Region (RR) of the LRT can be written as

$$|y - \frac{n}{2}| > c$$

where c is a constant which depends on the size of the RR.

14. A random sample of size n is to be used to test the null hypothesis that the parameter θ of an exponential population equals θ_0 against the alternative that it does not equal θ_0.

[a] Find an expression for the Likelihood Ratio statistics.

[b] Use the result of part [a] to show that the RR of the LRT can be written as

$$\overline{y} e^{-\overline{y}/\theta_0} < c$$

[c] Show that the RR is of the form $\{\overline{y} : \overline{y} > c_1 \text{ or } \overline{y} < c_2\}$

15. A random sample of size n from a normal population with unknown mean and variance is to be used to test the null hypothesis $\mu = \mu_0$ against the alternative $\mu \neq \mu_0$.

[a] Use the Maximum Likelihood Estimates of μ and σ^2 to show that the value of the Likelihood Ratio Statistics can be written in the form

$$\lambda = \left(1 + \frac{t^2}{n-1}\right)^{-n/2}$$

where $t = \frac{\overline{y} - \mu_0}{s/\sqrt{n}}$

[b] Show that the RR is of the form $t^2 > c$.

[c] Show that $c = (t_{\alpha/2})^2$

16. Let $Y_1, Y_2, \ldots Y_n$ be a random sample from the geometric probability function

$$p_Y(y|p) = (1-p)^{y-1} p \qquad (y = 1, 2, \ldots)$$

Find the Likelihood Ratio Statistic λ for testing the null $p = p_0$ against the alternative $p \neq p_0$

17. A random sample of size $n = 8$ is drawn from the uniform pdf, $f_Y(y|\theta) = 1/\theta \quad (0 < y < \theta)$ for the purpose of testing $H_0 : \theta = 2.0$ versus $H_A : \theta < 2.0$ at the $\alpha = 0.10$ level of significance.

[a] Find the likelihood ratio statistic λ.

[b] Find the rejection region for the Likelihood Ratio Test, when $\alpha = 0.10$