Crystal combinatorics from PBW bases[1]

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with John Claxton, Ben Salisbury and Adam Schultze

Loyola University Chicago

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[1] sildes available at http://webpages.math.luc.edu/∼ ptingley/

Outline

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- There is one such basis $B_{\mathbf{i}}$ for each expression $\mathbf{i}$ of w_0.

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- Can do some (pretty annoying but "elementary") linear algebra to show $\mathrm{span}_{\mathbb{Z}[q]}\{F^{(a)}_{\mathbf{i}\beta_k} F^{(b)}_{\mathbf{i}\beta_{k+1}} F^{(c)}_{\mathbf{i}\beta_{k+2}}\} = \mathrm{span}_{\mathbb{Z}[q]}\{F^{(a)}_{\mathbf{i}'\beta_k} F^{(b)}_{\mathbf{i}'\beta_{k+1}} F^{(c)}_{\mathbf{i}'\beta_{k+2}}\}$. $\square$

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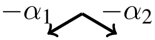

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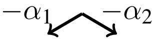

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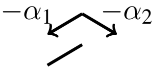

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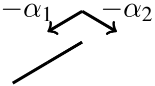

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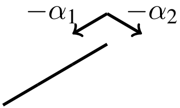

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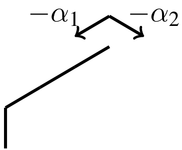

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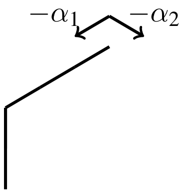

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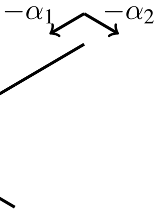

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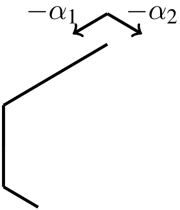

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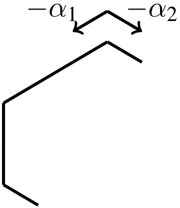

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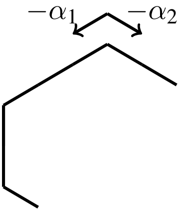

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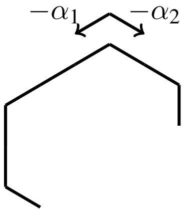

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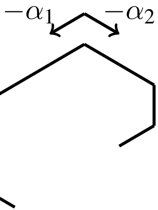

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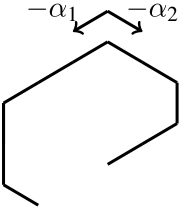

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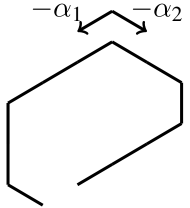

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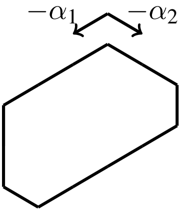

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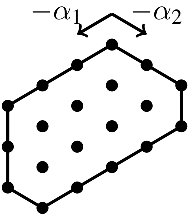

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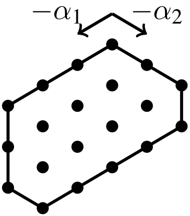

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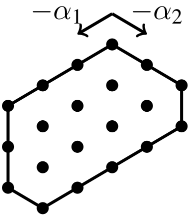

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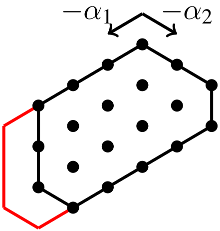

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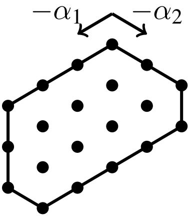

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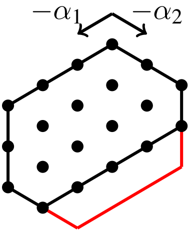

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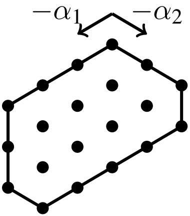

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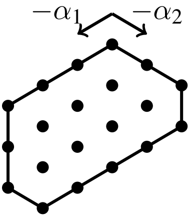

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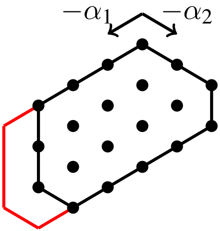

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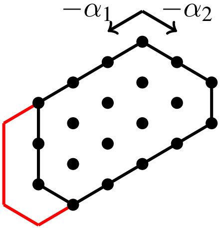

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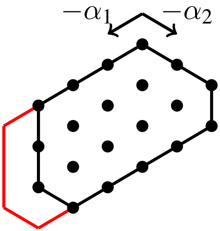

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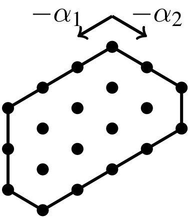

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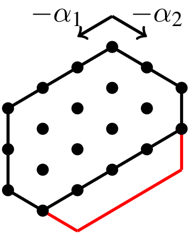

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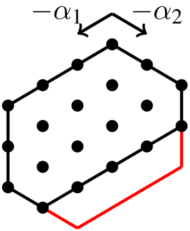

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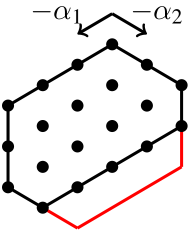

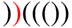

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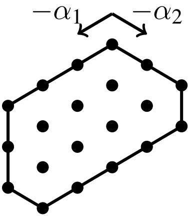

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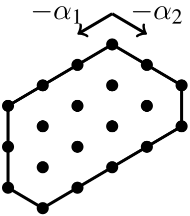

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Calculating crystal operators using braid moves: $\mathfrak{sl}_4$

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$s_1 \quad s_2 \quad s_3 \quad s_1 \quad s_2 \quad s_1$

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s_1	s_2	s_3	s_1	s_2	s_1
α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3

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	s_1	s_2	s_3	s_1	s_2	s_1
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$$\begin{array}{ccccccc}
 & s_1 & s_2 & s_3 & s_1 & s_2 & s_1 \\
 & \alpha_1 & (\alpha_1+\alpha_2) & (\alpha_1+\alpha_2+\alpha_3) & \alpha_2 & (\alpha_2+\alpha_3) & \alpha_3 \\
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\end{array}$$

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	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$

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$$
\begin{array}{lcccccc}
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	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_3^{(1)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_3^{(2)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(4)}$	$F_{32}^{(2)}$	$F_2^{(4)}$

Calculating crystal operators using braid moves: $\mathfrak{sl}_4$

	s_1	s_2	s_3	s_1	s_2	s_1
	α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3
e.g.	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$
f_3 :	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(3)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_3^{(1)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$\color{red}{F_3^{(2)}}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(4)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(2)}$	$F_{23}^{(4)}$	$F_3^{(2)}$

Calculating crystal operators using braid moves: $\mathfrak{sl}_4$

	s_1	s_2	s_3	s_1	s_2	s_1
	α_1	$(\alpha_1+\alpha_2)$	$(\alpha_1+\alpha_2+\alpha_3)$	α_2	$(\alpha_2+\alpha_3)$	α_3
e.g.	$F_1^{(2)}$	$F_{12}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(3)}$	$F_{23}^{(3)}$	$F_3^{(2)}$
f_3 :	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(3)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_3{}^{(1)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_3^{(1)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_3^{(2)}$	$F_1{}^{(2)}$	$F_{312}^{(3)}$	$F_{12}^{(1)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_3^{(4)}$	$F_{32}^{(2)}$	$F_2^{(4)}$
	$F_1^{(2)}$	$F_{12}{}^{(3)}$	$F_{123}^{(1)}$	$F_2^{(2)}$	$F_{23}^{(4)}$	$F_3^{(2)}$

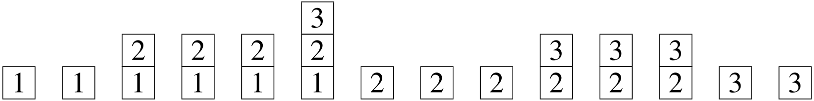
Calculating braid moves using segments/Kostant partitions

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$



Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

$$\begin{array}[b]{|c|}\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline\end{array}$$

$$\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 1\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 2\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline\end{array}\;
\begin{array}[b]{|c|}\hline 3\\\hline\end{array}$$

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

$$\begin{array}[b]{|c|}\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline\end{array}$$

f_3

$$\begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 1\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 2\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline\end{array}\ \begin{array}[b]{|c|}\hline 3\\\hline\end{array}$$

Calculating braid moves using segments/Kostant partitions

	$F_1^{(2)}$		$F_{12}{}^{(3)}$			$F_{123}^{(1)}$	$F_2^{(3)}$			$F_{23}^{(3)}$			$F_3^{(2)}$	
	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{3}$ $\boxed{2}$ $\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\boxed{3}$ $\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{2}$ $\boxed{1}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{3}$ $\boxed{2}$	$\boxed{1}$	$\boxed{1}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}{}^{(3)} \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

						3								
			2	2	2	2				3	3	3		
	1	1	1	1	1	1	2	2	2	2	2	2	3	3
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	3													
	2	2	2	2	3	3	3							
	1	1	1	1	2	2	2	1	1	2	2	2	3	3

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \qquad F_{12}(3) \qquad F_{123}^{(1)} \qquad F_2^{(3)} \qquad F_{23}^{(3)} \qquad F_3^{(2)}$$

	$\boxed{1}$	$\boxed{1}$	$\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\boxed{2}$	$\boxed{2}$	$\boxed{2}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\boxed{3}$	$\boxed{3}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{2}\\\boxed{1}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\boxed{1}$	$\boxed{1}$	$\begin{smallmatrix}\boxed{3}\\\boxed{2}\end{smallmatrix}$	$\boxed{2}$	$\boxed{2}$	$\boxed{3}$	$\boxed{3}$

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$$

						3								
			2	2	2	2				3	3	3		
	1	1	1	1	1	1	2	2	2	2	2	2	3	3
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	3													
	2	2	2	2	3	3	3			3				
	1	1	1	1	2	2	2	1	1	2	2	2	3	3

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(2)} \quad F_{23}^{(4)} \quad F_3^{(2)}$$

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$$

						3								
			2	2	2	2				3	3	3		
	1	1	1	1	1	1	2	2	2	2	2	2	3	3
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	3													
	2	2	2	2	3	3	3			3				
	1	1	1	1	2	2	2	1	1	2	2	2	3	3

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(2)} \quad F_{23}^{(4)} \quad F_3^{(2)}$$

Calculating braid moves using segments/Kostant partitions

	$F_1^{(2)}$		$F_{12}^{(3)}$			$F_{123}^{(1)}$	$F_2^{(3)}$			$F_{23}^{(3)}$			$F_3^{(2)}$	
	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 1\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 2\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$	$\begin{array}{\|c\|}\hline 3\\\hline\end{array}$
	$F_1^{(2)}$		$F_{12}^{(3)}$			$F_{123}^{(1)}$	$F_2^{(2)}$			$F_{23}^{(4)}$			$F_3^{(2)}$	

- Gives a bracketing rule as long as each α_i can be moved to the front with all 3-term moves involving α_i.

Calculating braid moves using segments/Kostant partitions

$F_1^{(2)}$ $F_{12}{}^{(3)}$ $F_{123}^{(1)}$ $F_2^{(3)}$ $F_{23}^{(3)}$ $F_3^{(2)}$

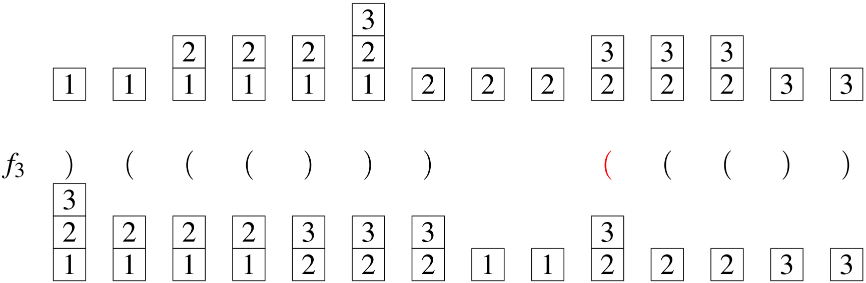
						3								
			2	2	2	2				3	3	3		
	1	1	1	1	1	1	2	2	2	2	2	2	3	3
f_3	)	(	(	(	)	)	)			(	(	(	)	)
	3													
	2	2	2	2	3	3	3			3				
	1	1	1	1	2	2	2	1	1	2	2	2	3	3

$F_1^{(2)}$ $F_{12}{}^{(3)}$ $F_{123}^{(1)}$ $F_2^{(2)}$ $F_{23}^{(4)}$ $F_3^{(2)}$

- Gives a bracketing rule as long as each α_i can be moved to the front with all 3-term moves involving α_i. There is a reduced expression with this property in all types except E_8

Calculating braid moves using segments/Kostant partitions

$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad F_2^{(3)} \quad F_{23}^{(3)} \quad F_3^{(2)}$$



$$F_1^{(2)} \quad F_{12}{}^{(3)} \quad F_{123}^{(1)} \quad {\color{blue}F_2^{(2)}} \quad {\color{blue}F_{23}^{(4)}} \quad F_3^{(2)}$$

- Gives a bracketing rule as long as each α_i can be moved to the front with all 3-term moves involving α_i. There is a reduced expression with this property in all types except E_8 (and F_4).

Calculating crystal operators using braid moves: type D_4

Calculating crystal operators using braid moves: type D_4

$$s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_2 \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_2 \quad s_3 \quad s_4$$

Calculating crystal operators using braid moves: type D_4

$$
\begin{array}{cccccccccccc}
s_1 & s_2 & s_3 & s_4 & s_2 & s_1 & s_2 & s_3 & s_4 & s_2 & s_3 & s_4 \\[1em]
1 & \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 3 \\ 2 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 4 \\ 2 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 3\,4 \\ 2 \\ 1 \end{smallmatrix} & \begin{smallmatrix} 2 \\ 3\,4 \\ 2 \\ 1 \end{smallmatrix} & 2 & \begin{smallmatrix} 4 \\ 2 \end{smallmatrix} & \begin{smallmatrix} 3 \\ 2 \end{smallmatrix} & \begin{smallmatrix} 3\,4 \\ 2 \end{smallmatrix} & 3 & 4
\end{array}
$$

Calculating crystal operators using braid moves: type D_4

s_1	s_2	s_3	s_4	s_2	s_1	s_2	s_3	s_4	s_2	s_3	s_4
1	2 1	3 2 1	4 2 1	3 4 2 1	2 3 4 2 1	2	4 2	3 2	3 4 2	3	4
		4 2 1	3 2 1							4	3

Calculating crystal operators using braid moves: type D_4

s_1	s_2	s_3	s_4	s_2	s_1	s_2	s_3	s_4	s_2	s_3	s_4
1	2 1	3 2 1	4 2 1	3 4 2 1	2 3 4 2 1	2	4 2	3 2	3 4 2	3	4
		4 2 1	3 2 1							4	3
								4	3 4 2	3 2	

Calculating crystal operators using braid moves: type D_4

s_1	s_2	s_3	s_4	s_2	s_1	s_2	s_3	s_4	s_2	s_3	s_4
1	2 1	3 2 1	4 2 1	3 4 2 1	2 3 4 2 1	2	4 2	3 2	3 4 2	3	4
		4 2 1	3 2 1							4	3
								4	3 4 2	3 2	
						4	4 2	2			

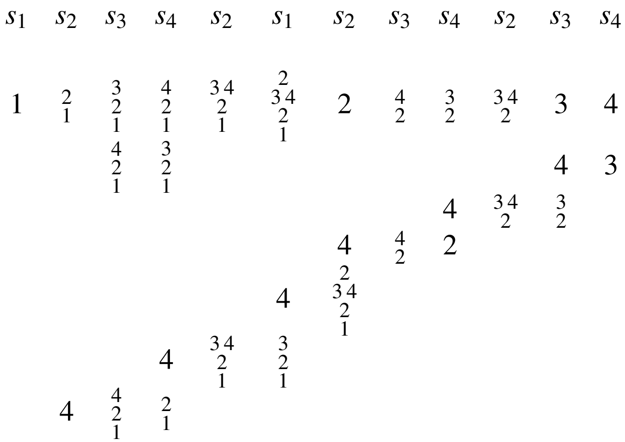
Calculating crystal operators using braid moves: type D_4

s_1	s_2	s_3	s_4	s_2	s_1	s_2	s_3	s_4	s_2	s_3	s_4
1	2 1	3 2 1	4 2 1	3 4 2 1	2 3 4 2 1	2	4 2	3 2	3 4 2	3	4
		4 2 1	3 2 1							4	3
								4	3 4 2	3 2	
						4	4 2	2			
					4	2 3 4 2 1					

Calculating crystal operators using braid moves: type D_4

s_1	s_2	s_3	s_4	s_2	s_1	s_2	s_3	s_4	s_2	s_3	s_4
1	2 1	3 2 1	4 2 1	3 4 2 1	2 3 4 2 1	2	4 2	3 2	3 4 2	3	4
		4 2 1	3 2 1							4	3
								4	3 4 2	3 2	
						4	4 2	2			
					4	2 3 4 2 1					
			4	3 4 2 1	3 2 1						

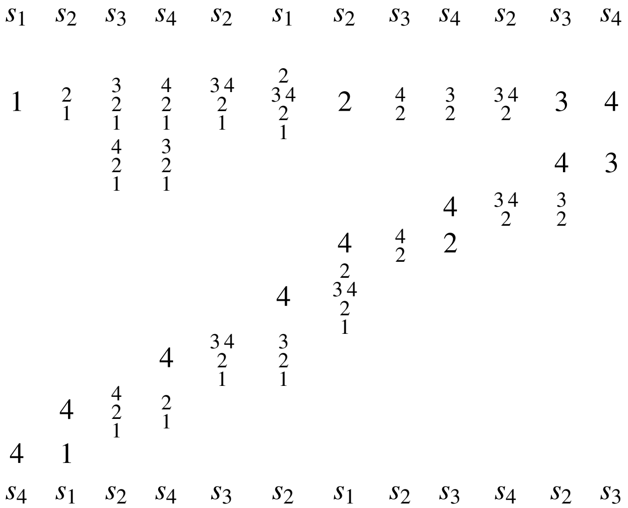
Calculating crystal operators using braid moves: type D_4



Calculating crystal operators using braid moves: type D_4

s_1	s_2	s_3	s_4	s_2	s_1	s_2	s_3	s_4	s_2	s_3	s_4
1	2 1	3 2 1	4 2 1	3 4 2 1	2 3 4 2 1	2	4 2	3 2	3 4 2	3	4
		4 2 1	3 2 1							4	3
								4	3 4 2	3 2	
						4	4 2	2			
					4	2 3 4 2 1					
			4	3 4 2 1	3 2 1						
	4	4 2 1	2 1								
4	1										

Calculating crystal operators using braid moves: type D_4



Calculating crystal operators using type D Kostant partition

Calculating crystal operators using type D Kostant partition

$$F^{(2)}_{1} \quad F^{(1)}_{\substack{2\\1}} \quad F^{(4)}_{\substack{3\\2\\1}} \quad F^{(2)}_{\substack{4\\2\\1}} \quad F^{(1)}_{\substack{3\,4\\2\\1}} \quad F^{(3)}_{\substack{2\\3\,4\\2\\1}} \quad F^{(3)}_{2} \quad F^{(1)}_{\substack{4\\2}} \quad F^{(2)}_{\substack{3\\2}} \quad F^{(1)}_{\substack{3\,4\\2}} \quad F^{(2)}_{3} \quad F^{(0)}_{4}$$

Calculating crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

Calculating crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

f_3

Calculating crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{3\,4\\2\\1}}^{(1)} \quad F_{\substack{2\\3\,4\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{3\,4\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \begin{matrix}2\\1\end{matrix} \quad \begin{matrix}3\\2\\1\end{matrix} \quad \begin{matrix}3\\2\\1\end{matrix} \quad \begin{matrix}3\\2\\1\end{matrix} \quad \begin{matrix}3\\2\\1\end{matrix} \quad \begin{matrix}4\\2\\1\end{matrix} \quad \begin{matrix}4\\2\\1\end{matrix} \quad \begin{matrix}3\,4\\2\\1\end{matrix} \quad \begin{matrix}2\\3\,4\\2\\1\end{matrix} \quad \begin{matrix}2\\3\,4\\2\\1\end{matrix} \quad \begin{matrix}2\\3\,4\\2\\1\end{matrix} \quad 2 \quad 2 \quad 2 \quad \begin{matrix}4\\2\end{matrix} \quad \begin{matrix}3\\2\end{matrix} \quad \begin{matrix}3\\2\end{matrix} \quad \begin{matrix}3\,4\\2\end{matrix} \quad 3 \quad 3$$

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$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{3\,4\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

Calculating crystal operators using type D Kostant partition

$$F^{(2)}_{1} \quad F^{(1)}_{\substack{2\\1}} \quad F^{(4)}_{\substack{3\\2\\1}} \quad F^{(2)}_{\substack{4\\2\\1}} \quad F^{(1)}_{\substack{3\,4\\2\\1}} \quad F^{(3)}_{\substack{2\\3\,4\\2\\1}} \quad F^{(3)}_{2} \quad F^{(1)}_{\substack{4\\2}} \quad F^{(2)}_{\substack{3\\2}} \quad F^{(1)}_{\substack{3\,4\\2}} \quad F^{(2)}_{3} \quad F^{(0)}_{4}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$) \quad) \quad) \quad) \quad (\quad) \quad (\quad (\quad) \quad) \quad (\quad (\quad (\quad) \quad (\quad) \quad) \quad (\ldots$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{3\,4\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

Calculating crystal operators using type D Kostant partition

$$F_{1}^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{34\\2\\1}}^{(1)} \quad F_{\substack{2\\34\\2\\1}}^{(3)} \quad F_{2}^{(3)} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{(2)} \quad F_{\substack{34\\2}}^{(1)} \quad F_{3}^{(2)} \quad F_{4}^{(0)}$$

$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{34\\2\\1} \quad \substack{2\\34\\2\\1} \quad \substack{2\\34\\2\\1} \quad \substack{2\\34\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{34\\2} \quad 3 \quad 3$$

$$) \quad) \quad) \quad) \quad (\quad) \quad (\quad (\quad) \quad) \quad (\quad (\quad (\quad) \quad (\quad) \quad) \quad (\ldots$$

$$f_3 \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{2\\1} \quad \substack{34\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\\2} \quad \substack{3\\2} \quad 2 \quad 2 \quad 2 \quad \substack{34\\2} \quad \substack{4\\2} \quad 3 \quad 3$$

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$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

$$\begin{array}{lcccccccccccccccccc}
 &) &) &) &) & (&) & (& (&) &) & (& (& (&) & (&) &) & (\ldots \\
f_3 & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{2\\1} & \substack{3\,4\\2\\1} & \substack{4\\2\\1} & \substack{4\\2\\1} & \substack{3\\2} & \substack{3\\2} & 2 & 2 & 2 & \substack{3\,4\\2} & \substack{4\\2} & 3 & 3 &
\end{array}$$

Calculating crystal operators using type D Kostant partition

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$$1 \quad 1 \quad \substack{2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{3\\2\\1} \quad \substack{4\\2\\1} \quad \substack{4\\2\\1} \quad \substack{3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad \substack{2\\3\,4\\2\\1} \quad 2 \quad 2 \quad 2 \quad \substack{4\\2} \quad \substack{3\\2} \quad \substack{3\\2} \quad \substack{3\,4\\2} \quad 3 \quad 3$$

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$$\begin{array}{lcccccccccccccccccc}
 &) &) &) &) & (&) & (& (&) &) & \color{red}{(} & (& (&) & (&) &) & (\ldots \\
f_3 & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{3\\2\\1} & \substack{2\\1} & \substack{34\\2\\1} & \substack{4\\2\\1} & \substack{4\\2\\1} & \substack{3\\2} & \substack{3\\2} & 2 & 2 & 2 & \substack{34\\2} & \substack{4\\2} & 3 & 3 &
\end{array}$$

$$F_1^{(2)} \quad F_{\substack{2\\1}}^{(1)} \quad F_{\substack{3\\2\\1}}^{(4)} \quad F_{\substack{4\\2\\1}}^{(2)} \quad F_{\substack{34\\2\\1}}^{(1)} \quad F_{\substack{2\\34\\2\\1}}^{(3)} \quad F_2^{\color{blue}{(2)}} \quad F_{\substack{4\\2}}^{(1)} \quad F_{\substack{3\\2}}^{\color{blue}{(3)}} \quad F_{\substack{34\\2}}^{(1)} \quad F_3^{(2)} \quad F_4^{(0)}$$

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4. The reduced expressions we need were all given by Littelmann in his paper “Cones, crystals and patterns," for kind of similar reasons. But his definition looks a little stronger than what we need, so we don’t currently have a proof that our construction probably doesn’t work in E_8.

Thanks!!!!!!