A MINUS SIGN THAT USED TO ANNOY ME BUT NOW I KNOW WHY IT IS THERE
(TWO CONSTRUCTIONS OF THE JONES POLYNOMIAL)

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Abstract. There are (at least) two well known constructions of link invariants. One uses skein theory: you resolve each crossing of the link as a linear combination of things that don't cross, until you eventually get a linear combination of links with no crossings, which you turn into a polynomial. The other uses quantum groups: you construct a functor from a topological category to some category of representations, in such a way that (oriented framed) links get sent to endomorphisms of the trivial representation, which are just rational functions. Certain instances of these two constructions give rise to essentially the same invariants, but when one carefully matches them there is a minus sign that seems out of place. We will discuss exactly how the constructions match up in the case of the Jones polynomial, and where the minus sign comes from.

Contents

1. Introduction

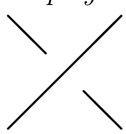
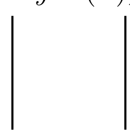
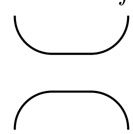
These notes are an expanded version of a talk given at the university of Queensland in Brisbane Australia, on Nov. 13, 2008. They are largely expository, and most of the content can be found in, for instance, [O, Appendix H]. The main difference here is that we use the non-standard ribbon element introduced in [ST]. We feel this makes the correspondence between the two constructions somewhat more transparent, mainly because it allows us to introduce a functor from framed but unoriented links to the category of representations of $U_q(\mathfrak{g})$. In the usual approach, the Kauffman bracket is calculated for unoriented framed links, while the quantum group invariants must be calculated for oriented framed links.

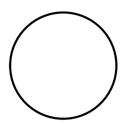
The non-standard ribbon we use exists in many cases beyond $U_q(\mathfrak{sl}_2)$. For instance, a very similar discussion to the following can be used to relate the type C quantum group knot invariants with a specialization of the Kauffman polynomial.

2. The Kauffman bracket and $U_q(\mathfrak{sl}_2)$ knot invariants

The following, up to a possible change in the variable q, is the well known construction of the Kauffman bracket.

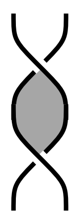
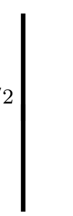
Definition 1. *Take a link diagram (i.e. A link drawn as a curve is the plan with under and over crossings). Simplify it using the following relations until the result is a polynomial in q and q^{-1}. That polynomial, denoted by $K(L)$, is the Kauffman bracket of the link diagram.*

(i)  $= q^{1/2}$  $+ \; q^{-1/2}$ 

(ii)  $= -q - q^{-1}$

(iii) *If two tangle diagrams are disjoint, their Kauffman brackets multiply.*

The Kauffman bracket is not an isotopy invariant of links, but is instead an invariant of framed links (that is, links tied out of orientable flat ribbons), where the framing is in the direction of the page. One can allow twists of the framing to occur in the diagram if one introduces the following extra relation (here both sides represent a single framed string):

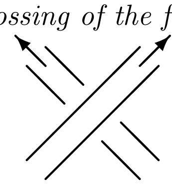
 $= -q^{3/2}$ 

regardless of orientation (note that the direction of the twist does matter).

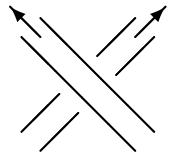
Theorem 2. *(see* [O, Theorem 1.10]*) The Kauffman bracket as calculated using the above three relations is an invariant of framed links.* □

Having to work with famed links instead of ordinary links is somewhat annoying. This is usually fixed as follows.

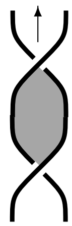
Definition 3. *An over-crossing is a crossing of the form*



An undercrossing is a crossing of the form



A positive full twist is a twist of the form



The writhe of a link diagram L, denoted by $w(L)$, is defined to the the number of over crossings minus the number of under crossings plus the number of positive full twists.

Comment 4. Here we have drawn every component with it's framing. Usually, unless we need to explicitly show a twist, we will just draw lines, and use the convention that these stand for ribbons lies flat on the page. This is often referred to as the "blackboard framing".

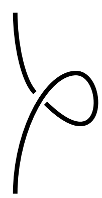
The following is one of the more fundamental theorems in knot theory. It says that the Jones polynomial is an invariant of framed links.

Theorem 5. *(see* [O, Theorem 1.5]*) Let L be any link. Then the Jones polynomial,*

$$J(L) := (-q^{3/2})^{w(L)} K(L), \tag{1}$$

is independent of the framing. Hence $J(L)$ is an isotopy invariant of oriented (but not framed) links.

This is normally stated in terms of link diagrams, not framed links. The result for framed links follows by noticing that the positive full twist from Definition 3 is isotopic to



with the black board framing.

Comment 6. It is straightforward to see that overcrossings are sent to overcrossings if both orientations are reversed. So, the orientations only affect the Jones polynomial if there are at least 2 components.

3. The quantum group construction of the Jones polynomial

3.1. **The quantum group $U_q(\mathfrak{sl}_2)$ and it's representations.** $U_q(\mathfrak{sl}_2)$ is an infinite dimensional algebra that is related to the Lie-algebra $\mathfrak{sl}_2$ of 2×2 matrices with trace zero. It is the algebra over the field of rational functions $\mathbb{C}(q)$ generated by E, F, K and K^{-1} subject to the relations

$$KK^{-1} = 1 \tag{2}$$

$$KEK^{-1} = q^2 E \tag{3}$$

$$KFK^{-1} = q^{-2} F \tag{4}$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{5}$$

It has a finite dimensional representation V_n for each integer n which we now describe. Introduce the quantum integers

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}. \tag{6}$$

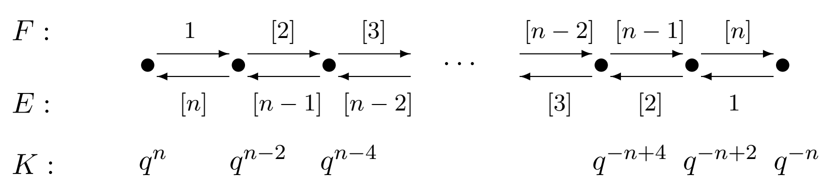
Then V_n is the $n+1$-dimensional with basis $v_n, v_{n-2}, \cdots, v_{-n+2}, v_{-n}$, where the action of E, F and K are given by

$$E(v_{-n+2j}) = \begin{cases} [j+1] v_{-n+2j+2} & \text{if } 0 \le j < n \\ 0 & \text{if } j = n. \end{cases} \tag{7}$$

$$F(v_{n-2j}) = \begin{cases} [j+1] v_{n-2j-2} & \text{if } 0 \le j < n \\ 0 & \text{if } j = n. \end{cases} \tag{8}$$

$$K(v_k) = q^k v_k. \tag{9}$$

This can be expressed by the following diagram:



We briefly discuss more structure on this category of representations. There is a tensor product on representations of $U_q(\mathfrak{sl}_2)$, where the action on $a \otimes b \in A \otimes B$ is given by

$$E(a \otimes b) = Ea \otimes Kb + 1 \otimes Eb \tag{10}$$

$$F(a \otimes b) = Fa \otimes b + K^{-1} \otimes Fb \tag{11}$$

$$K(a \otimes b) = Ka \otimes Kb. \tag{12}$$

For each pair of representations A and B, $A \otimes B$ is always isomorphic to $B \otimes A$. For all A, B, there is a chosen isomorphism

$$\sigma^{br}_{A,B} : A \otimes B \to B \otimes A. \tag{13}$$

This is called the braiding, and has a standard (but at times unenlightening) definition, which can be found in, for example [CP]. We omit this because Theorem 24 below gives an alternative definition which we find more appealing. Here we only ever apply the braiding to the standard 2-dimensional representation of $U_q(\mathfrak{sl}_2)$, so we can use the following:

Definition 7. *Let V be the 2 dimensional representation of $U_q(\mathfrak{sl}_2)$. Use the standard basis $\{v_1 \otimes v_1, v_{-1} \otimes v_1, v_1 \otimes v_{-1}, v_{-1} \otimes v_{-1}\}$ for $V \otimes V$. Then $\sigma^{br} : V \otimes V \to V \otimes V$ is given by the matrix*

$$\sigma^{br} = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

There is a standard action of $U_q(\mathfrak{sl}_2)$ on the dual vector space to V_n. This is defined using the "antipode" S. This is the algebra anti-automorphism defined on generators by:

$$\begin{cases} S(E) = -EK^{-1} \\ S(F) = -KF \\ S(K) = K^{-1} \end{cases} \tag{14}$$

Then $U_q(\mathfrak{sl}(2)$ acts on V_n^* by: For $\hat{v} \in V_n^*$ and $X \in U_q(\mathfrak{g})$, $X(\hat{v})$ is the element of V_n^* defined by

$$X\hat{v}(w) := \hat{v}(S(X)w), \tag{15}$$

for all $w \in V_n$.

It turns out that V_n is always isomorphic to V_n^*. The following is one possible isomorphism for the case of the standard (two dimensional) representation $V = V_1$.

Example 8. An isomorphism between the standard representation of $U_q(\mathfrak{sl}_2)$ and it's dual. Let v_1, v_{-1} be the basis for V. For $i = \pm 1$, let $\hat{v}_i$ be the element of V^* defined by

$$\hat{v}_i(v_j) = \delta_{i,j}. \tag{16}$$

Calculating using the above definition, the action of $U_q(\mathfrak{sl}_2)$ on V^* is given by

$$\hat{v}_{-1} \underset{-q}{\overset{-q^{-1}}{\rightleftarrows}} \hat{v}_1 \tag{17}$$

where the top arrow shows the action of F and the bottom arrow shows the action of E.

This turn out to be isomorphic to V itself. It will be useful for us to choose an isomorphism (this choice is arbitrary, but we need to make one).

Definition 9. *Let $f : V \to V^*$ be the map*

$$\begin{cases} f(v_1) = \hat{v}_{-1} \\ f(v_{-1}) = -q^{-1}\hat{v}_1 \end{cases} \tag{18}$$

One can easily check that f is an isomorphism.

Comment 10. If one sets $q = 1$, the category of representations described above is exactly the category of finite dimensional representations of the usual Lie algebra $\mathfrak{sl}_2$, where one identifies E and F with generators of $\mathfrak{sl}_2$ by

$$E \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \frac{K - K^{-1}}{q - q^{-1}} \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{19}$$

For us, this will be sufficient justification for thinking of $U_q(\mathfrak{sl}_2)$ as related to ordinary $\mathfrak{sl}_2$.

Comment 11. Notice that in the above K itself does not correspond to anything in $\mathfrak{sl}_2$, and acts as the identity at $q = 1$. $U_q(\mathfrak{sl}_2)$ actually has some other finite dimensional representations where K does not act as the identity at $q = 1$. We have restricted to the so called "type 1" representations. The other representations rarely appear in the theory.

3.2. **Ribbon elements and quantum traces.** The following constructions can be found in [CP, Chapter 4]. The main difference here is that we work with two possible ribbon elements throughout. Each satisfies the definition of a ribbon element in [CP]. For this reason we also have two different quantum traces, and two different co-quantum traces.

Definition 12. *The standard ribbon element Q_s is a central element in a certain completion of $U_q(\mathfrak{sl}_2)$, defined by the fact that it acts on each V_n as multiplication by the scalar $q^{-n^2/2-n}$. The "half-twist" ribbon element Q_t acts on V_n as multiplication by the scalar $(-1)^n q^{-n^2/2-n}$.*

Definition 13. *Let g_s be the operators which acts on $v_{n-2j} \in V_n$ as multiplication by $q^{(n-2j)/2}$. This is the standard group-like element used in defining quantum traces.*

Let g_t be the operators which acts on $v_{n-2j} \in V_n$ as multiplication by $(-1)^n q^{(n-2j)/2}$. This is a twisted group-like element which can also be used to define quantum traces.

Comment 14. The group like element is related to the ribbon element by

$$g = v^{-1}\mu(S \otimes Id)R_{21}, \tag{20}$$

where μ means multiplication and R_{21} is R applied to the tensor products in the reverse order. This fits into the standard theory of ribbon Hopf algebras as described in, for example, [CP, Chapter 4.2C].

Definition 15. *(see* [CP, Definition 4.29] *and* [O, Section 4.2]*) Define the following maps:*

(i) *ev is the evaluation map $V^* \otimes V \to F$.*
(ii) *qtr_{Q_s} is the standard quantum trace map $V \otimes V^* \to F$ defined by, for $\phi \in End(V) = V \otimes V^*$, $qtr_{Q_s}(\phi) = trace(\phi \circ g_s)$.*
(iii) *qtr_{Q_t} is the "half-twist" quantum trace map $V \otimes V^* \to F$ defined by, for $\phi \in End(V) = V \otimes V^*$, $qtr_{Q_t}(\phi) = trace(\phi \circ g_t)$.*
(iv) *$coev$ is the coevaluation map $F \to V \otimes V^*$ defined by $coev(1) = Id$, where Id is the identity map in $End(V) = V \otimes V^*$.*
(v) *$coqtr_{Q_s}$ is the standard quantum cotrace map $F \to V^* \otimes V$ defined by $coqtr_{Q_s}(1) = (g_s \otimes 1) \circ Flip \circ coev(1)$.*
(vi) *$coqtr_{Q_t}$ is the "half-twist" quantum cotrace map $F \to V^* \otimes V$ defined by $coqtr_{Q_t}(1) = (g_t \otimes 1) \circ Flip \circ coev(1)$.*

Comment 16. It is not approri clear that these maps are all morphisms of $U_q(\mathfrak{sl}_2)$ representations, since Flip is not generally a morphism. It turns out that the compositions used above are all morphisms. See [ST] for details.

Comment 17. In order to do calculations, it is often usefull to express these maps is coordinates. So, fix $f \in V^*$, $v \in V$, and $\{e^i\}$, $\{e_i\}$ be dual basis for V^* and V. Then the maps qtr_Q

and $coqtr_Q$ are given by

$$qtr_Q(v \otimes f) = f(gv) \tag{21}$$

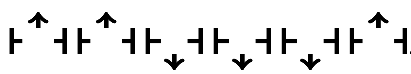
$$coqtr_Q(1) = \sum_i e^i \otimes g^{-1} e_i. \tag{22}$$

One can choose Q to be either Q_s or Q_t, and then one must use the grouplike element g_s or g_t accordingly.

3.3. **A topological category of ribbons with half-twists.** The construction of the $U_q(\mathfrak{sl}_2)$ quantum group knot invariants is possible because there is a functor from a certain topological category to the category of representations $U_q(\mathfrak{sl}_2)$. We now describe the relevant category, and this functor.

Definition 18. *$\mathcal{RIBBON}$ (topological ribbons) is the category defined by:*

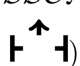
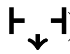
- *Objects in $\mathcal{RIBBON}$ consist of a finite number of disjoint closed intervals on the real line each directed either up or down. These objects are considered up to isotopy of the real line. For example:*



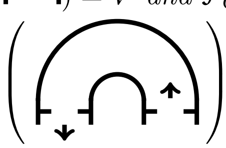
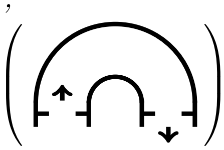
- *A morphism between two objects $A, B \in \mathcal{RIBBON}$ consist of 'tangles of orientable, directed ribbons' in $\mathbb{R}^2 \times I$, whose 'loose ends" are exactly $(A, 0, 0) \cup (B, 0, 1) \subset \mathbb{R} \times \mathbb{R} \times I$, such that the direction (up or down) of each interval in $A \cup B$ agrees with the direction of the ribbon whose end lies at that interval. These are considered up to isotopy.*
- *Composition of two morphisms is given by stacking them on top of each other, and them shrinking the vertical axis by a factor of two. Note that we read our diagrams bottom to top.*

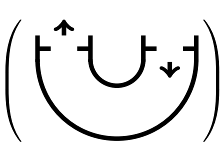
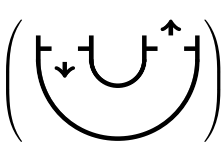
3.4. **The functor.** The following is the main ingredient to the quantum group construction of knot invariants. It holds in much greater generality than stated here, which allows for the construction of a great many invariants.

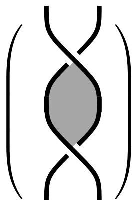
Theorem 19. *(see* [CP, Theorem 5.3.2]*) Let V be the standard 2 dimensional representation of $U_q(\mathfrak{sl}_2)$. For each ribbon element Q (here Q_s or Q_t), there is a unique monoidal functor $\mathcal{F}_Q$ from $\mathcal{RIBBON}$ to $U_q(\mathfrak{gl}_2)$-rep such that*

(i) $\mathcal{F}_Q($  $) = V$ *and* $\mathcal{F}_Q($  $) = V^*$,

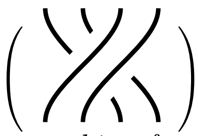
(ii)

$\mathcal{F}_Q$  $= ev$, $\mathcal{F}_Q$  $= qtr_Q$,

$\mathcal{F}_Q$  $= coev$, $\mathcal{F}_Q$  $= coqtr_Q$,

(iii) $\mathcal{F}_Q$  $= Q$,

thought of as a morphism from V to V or from V^* to V^*, depending on the orientation.

(iv) $\mathcal{F}_Q$  $= Flip \circ R$

as a morphism from the tensor product of the bottom two objects to the tensor product of the top two objects, regardless of orientation.

Let L be a link. The image of L under the above functor is a morphism from $\mathbb{C}(q)$ to $\mathbb{C}(q)$ in $\mathbf{rep} - U_q(\mathfrak{sl}_2)$, which is just a rational function in q. We denote this by

$$I_Q(L) := \mathcal{F}_Q(L), \text{ considered as a function of } q. \tag{23}$$

In fact, $I_Q(L)$ is always a polynomial in $q^{1/2}$ and $q^{-1/2}$.

Theorem 20. *(see* [O, Theorem 4.19] *) For any framed link L, we have $I_{Q_s}(L) = (-1)^{n(L)} \langle L \rangle$, where $n(L)$ is some integer depending on L.*

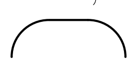
The minus sign in Theorem 20 is clearly annoying (and is the reason for the title of these notes). Theorem 22 below shows how this annoyance is removed by using Q_t in place of Q_s.

Comment 21. Given a diagram of L, $n(L)$ is the number of right going cups, plus the number of right going caps, plus the number of full (360 degree) twists of the framing. Alternatively, as is done in [O], $n(L)$ can be defined in terms of the number of components of L and the "framing number" of each component.

4. Matching the two constructions, in the case when the ribbon element is Q_t

We will now show how the skein relations used in defining the Kauffman bracket can be explained using the quantum group formulation. This section is similar to [O, Appendix H], although the presentation is simplified by use the non-standard ribbon element Q_t throughout.

The Kauffman bracket is calculated for unoriented link diagrams, so we will need an interpretation of an unoriented link diagram as a morphism in the category of $U_q(\mathfrak{g})$ representations. For this, all the strands will be interpreted as the two dimensional representation V (as opposed to V^*). We will then interpret each feature in the knot diagram as a morphism between the appropriate tensor powers of V For instance,



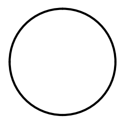
should be interpreted as a morphism from $V \otimes V$ to $\mathbb{C}$. This is as opposed to oriented caps, which represent morphisms $V^* \otimes V \to \mathbb{C}$ or $V \otimes V^* \to \mathbb{C}$. To do this, notice that in the case we are considering, V is isomorphic to V^*. Choose, once and for all, an isomorphism $f : V \to V^*$ (for instance, one can use the isomorphism from Definition 9). Associate morphisms of $U_q(\mathfrak{sl}_2)$ representations to unoriented caps and cups by:

(24)  $\longrightarrow ev \circ (f \otimes \mathrm{Id}) = qtr_{Q_t} \circ (\mathrm{Id} \otimes f) : V \otimes V \to \mathbb{C}(q)$

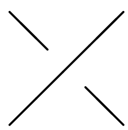
(25)  $\longrightarrow (\mathrm{Id} \otimes f^{-1}) \circ coev = (f^{-1} \otimes \mathrm{Id}) \circ coqtr_{Q_t} : \mathbb{C}(q) \to V \otimes V.$

In both of these equations, equality of the two expressions on the right can easily be verified. Note that quantum trace and co-quantum trace in fact depend on a choice of ribbon element, and the subscript indicates that we are using the ribbon element Q_t. If one used Q_s instead of Q_t, the expressions are off by a minus sign, and this construction does not work.

By direct calculation, one then sees that

(26)  $\longrightarrow$ multiplication by $-q - q^{-1}$.

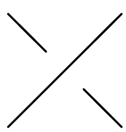
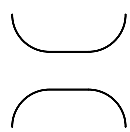
As is standard, we will represent the braiding diagrammatically by a crossing:

(27)  $= \sigma^{br}$

One can easily check, again by direct computation, that:

$$\sigma^{br} = q^{1/2} Id + q^{-1/2} (\mathrm{Id} \otimes f^{-1}) \circ coev \circ qtr_{Q_t} \circ (\mathrm{Id} \otimes f) : V \otimes V \to V \otimes V. \tag{28}$$

Expressing this equation diagrammatically, using the above notation, we see that:

(29)  $= \quad q^{1/2}$ $+ \quad q^{-1/2}$ 

But this is exactly the skein relation used to calculate the Kauffman bracket! Along with Equation 26, this implies that $\mathcal{F}_{Q_t}$ of an unoriented link gives the Kauffman bracket. From this it of course follows that one obtains an invariant of unoriented framed links, which we will call $I^{unoriented}(L)$. We now prove that this in in fact the same invariant as $I_{Q_t}(L)$ for any orientation of L:

Theorem 22. *Let L be a framed link. Then $I_{Q_t}(L)$ is independent of the chosen orientation. Furthermore, it agrees with $I^{unoriented}(L)$, and hence also with the Kauffman bracket*

Proof. Take any orientations of L. Add $f \circ f^{-1}$ on every component that is oriented down. This clearly doesn't change the morphism. By the naturality of the braiding σ^{br},

$$(1 \otimes f)\sigma^{br} = \sigma^{br}(f \otimes 1). \tag{30}$$

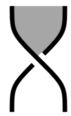
This allows one to pull all the f and f^{-1} through crossings until they are right next to cups and caps. But now you are exactly calculating $I^{unoriented}(L)$. □

5. Another advantage: the half twist

The non-standard ribbon element Q_t has a square root, which we will call X^{-1}, that has an interesting topological interpretation.

Definition 23. *Let X act on V_n by $X v_{n-2j} = i^n q^{n^2/4+n/2} v_{-n+2j}$, where i is the complex number i.*

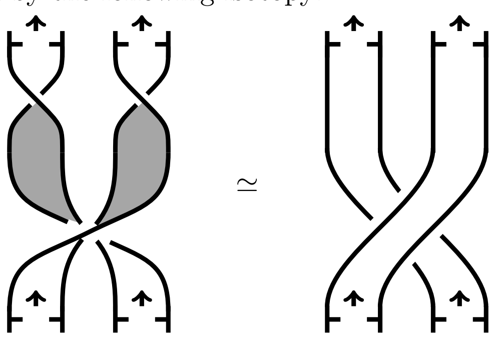
Then one can easily check that $X^{-2} = Q_t$. Thus one is led to try and extend the definition of the functor $\mathcal{F}_{Q_t}$ from Theorem 19 in such a way that

(31) $$\mathcal{F}_{Q_t}\left(\right) = X^{-1}.$$


The following gives further indications that such a generalized functor should exist. It is a specialization of a result of Kirilov-Reshetikhin[KR, Theorem 3] and Levendorskii-Soibelman, [LS, Theorem 1]. See [KT, Definition 4.6 and Corollary 8.4] for this exact statement. Since we have not defined the braiding σ^{br}, one could take this as the definition.

Theorem 24. $\sigma^{br} = (X^{-1} \otimes X^{-1}) \circ \mathit{Flip} \circ \Delta(X)$.

This can be interpreted by the following isotopy:



X should correspond to twisting the ribbon 180 degrees, and Flip $\circ\, \Delta(X)$ to twisting both at once by 180 degree, as on the bottom of the left side above.

This extended functor is defined precisely in [ST], resulting in a functor from a larger category where ribbons are allowed to twist by 180 degrees, not just by 360 degrees (although Mobius bands are still not allowed). It is only possible to extend $\mathcal{F}_{Q_t}$, not $\mathcal{F}_{Q_s}$. One advantage of this extension over the original functor is that, since both σ^{br} and Q_t are constructed in term of the "half-twist" X, there is in some sense one less elementary feature to consider. Morphisms in the resulting category are as in Figure 1 (ignoring the letters). Notice that elementary objects come in both shaded and unshaded versions.

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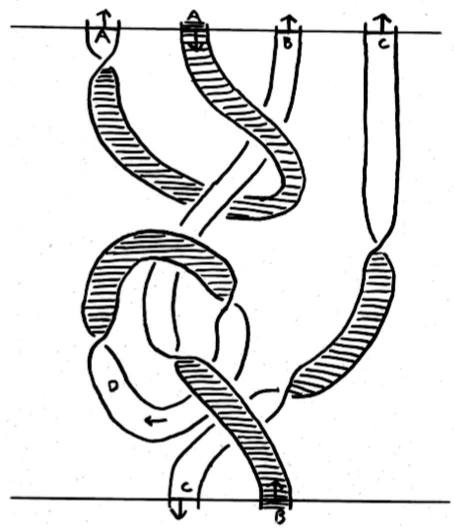

FIGURE 1. A morphism in the topological category of ribbons with half twists