CONSTRUCTING THE R-MATRIX FROM THE QUASI R-MATRIX.

PETER TINGLEY

1. Introduction

These notes are mainly a companion to [T], and we refer to that paper for any notation which we do not define here. In particular, in [T] we needed the following statement about the universal R-matrix for $U_q(\mathfrak{g})$:

Proposition 1.1. *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Then $U_q(\mathfrak{g})$ has a unique universal R-matrix of the form*

$$R = A\Big(1 \otimes 1 + \sum_{\substack{\text{positive integral} \\ \text{weights } \beta \text{ (with} \\ \text{multiplicity)}}} X_\beta \otimes Y_\beta\Big), \tag{1}$$

where X_β has weight β, Y_β has weight $-\beta$, and for all $v \in V$ and $w \in W$, $A(v \otimes w) = q^{(\mathrm{wt}(v), \mathrm{wt}(w))}$.

When $\mathfrak{g}$ is of finite type, this follows quite easily from, for example, [CP, Theorem 8.3.9]. In the case of a general symmetrizable Kac-Moody algebra, the only source we know is [L, Chapter 4]. However, in [L], they use the so called quasi R-matrix in place of the universal R-matrix. In the following we show how to recover Proposition 1.1 from their statements.

2. The conversion

We have kept the statements out of [L] as close to the original as possible, but have made minor modifications to avoid notational confusion. In particular, we have changed the notation for the components of the quasi R-matrix from Θ_γ to M^γ. Also, Δ in [L] is Δ^{op} in our notation (see below). We state by recalling the usual definition of a universal R-matrix.

Definition 2.1. *A universal R-matrix is an element R of $\widetilde{U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})}$ such that $\sigma^{br}_{V,W} := Flip \circ R$ is a braiding on the category of $U_q(\mathfrak{g})$ representations. Equivalently, an element R is a universal R-matrix if it satisfies the following three conditions*

(i) *For all $u \in U_q(\mathfrak{g})$, $R\Delta(u) = \Delta^{op}(u)R$.*
(ii) $(\Delta \otimes 1)R = R_{13}R_{23}$, *where R_{ij} mean R placed in the i and j^{th} tensor factors.*
(iii) $(1 \otimes \Delta)R = R_{13}R_{12}$.

Definition 2.2. *The bar-coproduct $\bar{\Delta}$ is defined by*

$$\begin{cases} \bar{\Delta}E_i &= E_i \otimes K_i^{-1} + 1 \otimes E_i \\ \bar{\Delta}F_i &= F_i \otimes 1 + K_i \otimes F_i \\ \bar{\Delta}K_i &= K_i \otimes K_i \end{cases}$$

Definition 2.3. *The opposite coproduct is $\Delta^{op} := Flip \circ \Delta$.*

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Definition 2.4. *The bar-opposite coproduct is* $\bar{\Delta}^{op} := Flip \circ \bar{\Delta}$.

We now state the main existence theorem from [L] for the quasi R-matrix.

Theorem 2.5. [L, Theorem 4.1.2 and Proposition 4.2.2] *There is a unique family of elements* $M^\gamma \in U_q(\mathfrak{g})^-_\gamma \otimes U_q(\mathfrak{g})^+_\gamma$ *with* $M^0 = 1 \otimes 1$ *and such that the "quasi R-matrix"* $M := \sum_\gamma M^\gamma$ *satisfies, for all* $u \in U_q(\mathfrak{g})$, $\Delta^{op}(u)M = M\bar{\Delta}^{op}(u)$. *Furthermore,*

(i) $(\Delta^{op} \otimes 1)(M) = \sum_{\gamma', \gamma'' \in P^+} M_{23}^{\gamma'}(1 \otimes K_{-\gamma''} \otimes 1)M_{13}^{\gamma''}$.
(ii) $(1 \otimes \Delta^{op})(M) = \sum_{\gamma', \gamma'' \in P^+} M_{12}^{\gamma'}(1 \otimes K_{\gamma''} \otimes 1)M_{13}^{\gamma''}$.

Here M_{ij}^γ *means* M^γ *in the* i *and* j^{th} *factor of the tensor product (tensored with 1 in the other positions).*

In order to derive Proposition 1.1 from Theorem 2.5, we need some terminology and a few lemmas.

Definition 2.6. *Let* J *be the operator which acts on* V *by* $Jv = q^{(\mathrm{wt}(v), \mathrm{wt}(v))/2 + (\mathrm{wt}(v), \rho)}v$. *Let* C_J *be the algebra automorphism of* $U_q(\mathfrak{g})$ *defined by*

$$\begin{cases} C_J(E_i) = K_i E_i \\ C_J(F_i) = F_i K_i^{-1} \\ C_J(K_H) = K_H \end{cases} \tag{2}$$

Comment 2.7. It is a simple calculation to show that C_J actually is an algebra automorphism.

Comment 2.8. We will use the notation $\Delta(J)$ to denote J acting on a tensor product.

Lemma 2.9. *J, C_J and the element A from Proposition 1.1 have the following properties:*

(i) $A = (J^{-1} \otimes J^{-1})\Delta(J) = \Delta(J)(J^{-1} \otimes J^{-1})$, *where, as in Proposition 1.1,* $A(v \otimes w) = q^{(\mathrm{wt}(v) \otimes \mathrm{wt}(w))} v \otimes w$.
(ii) *For all* $u \in U_q(\mathfrak{g})$, $(C_J \otimes C_J)\Delta(u) = \bar{\Delta}^{op}(C_J(u))$.
(iii) *The following diagram commutes*

$$\begin{tikzcd} V \arrow[r, "J"] \arrow[d, phantom, "\circlearrowright"] & V \arrow[d, phantom, "\circlearrowright"] \\ U_q(\mathfrak{g}) \arrow[r, "C_J"] & U_q(\mathfrak{g}). \end{tikzcd}$$

(iv) *For all* $u \in U_q(\mathfrak{g})$, $A^{-1}\Delta(u) = \bar{\Delta}^{op}(u)A^{-1}$.

Proof. For (i), fix weight vectors v and w, and simply do all three calculations on $v \otimes w$.

For (ii) one needs only check the equality on the generators E_i, F_i and K_i. Each is a straightforward calculation.

For (iii), pick a weight vector $v \in V$, and a generator $X = E_i, F_i$ or K_i. It is then a simple calculation to check that $J(X(v)) = C_J(X)(J(v))$. For example, if $X = E_i$, then

$$\begin{aligned} J(E_i v) &= q^{(\mathrm{wt}(E_i v), \mathrm{wt}(E_i v))/2 + (\mathrm{wt}(E_i v), \rho)} E_i v = q^{(\mathrm{wt}(v)+\alpha_i, \mathrm{wt}(v)+\alpha_i)/2 + (\mathrm{wt}(v)+\alpha_i, \rho)} E_i v = \\ &= q^{(\mathrm{wt}(v)+\alpha_i, \alpha_i)} E_i q^{(\mathrm{wt}(v), \mathrm{wt}(v))/2 + (\mathrm{wt}(v), \rho)} v = K_i E_i J(v) = C_J(E_i) J(v). \end{aligned} \tag{3}$$

Here we have used the fact that $(\alpha_i, \rho) = (\alpha_i, \alpha_i)/2$.

For (iv): By parts (i), (ii) and (iii), the following diagram commutes:

$$\begin{tikzcd}
U \otimes V \arrow[r, "\Delta(J)^{-1}"] \arrow[rr, bend left, "A^{-1}"] \arrow[d, phantom, "\circlearrowright"] & U \otimes V \arrow[r, "J\otimes J"] \arrow[d, phantom, "\circlearrowright"] & U \otimes V \arrow[d, phantom, "\circlearrowright"] \\
U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) & U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \arrow[r, "C_J\otimes C_J"] & U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g}) \\
U_q(\mathfrak{g}) \arrow[u, "\Delta"] \arrow[r, "C_J^{-1}"] & U_q(\mathfrak{g}) \arrow[u, "\Delta"] \arrow[r, "C_J"] & U_q(\mathfrak{g}). \arrow[u, "\bar{\Delta}^{op}"]
\end{tikzcd}$$

The equality is given by fixing $u \in U_q(\mathfrak{g})$, and following the diagram around in the two directions, recalling that $A^{-1} = (J \otimes J)\Delta(J)^{-1}$. $\square$

Proposition 2.10. *$R := MA^{-1}$ is a universal R-matrix.*

Proof. It suffices to check the three conditions in Definition 2.1.

(i): Fix $u \in U_q(\mathfrak{g})$. Then

$$R\Delta(u) = MA^{-1}\Delta(u) \tag{4}$$
$$= M\bar{\Delta}^{op}(u)A^{-1} \tag{5}$$
$$= \Delta^{op}(u)MA^{-1}. \tag{6}$$

Here (5) follows from Lemma 2.9 part (iv), and (6) follows from Theorem 2.5.

Part (ii): Let s_{12} be the permutation the interchanges the first and second tensor factors.

$$(\Delta \otimes 1)(R)(u \otimes v \otimes w) \tag{7}$$
$$=(\Delta \otimes 1)(MA^{-1})(u \otimes v \otimes w) \tag{8}$$
$$=s_{12}(\Delta^{op} \otimes 1)(MA^{-1})s_{12}(u \otimes v \otimes w) \tag{9}$$
$$=s_{12}(\Delta^{op} \otimes 1)Mq^{-(\mathrm{wt}(u)+\mathrm{wt}(v),\mathrm{wt}(w))}(v \otimes u \otimes w) \tag{10}$$
$$=s_{12}\sum_{\gamma',\gamma''\in Q^+} M_{23}^{\gamma'}(1 \otimes K_{-\gamma''} \otimes 1)M_{13}^{\gamma''}q^{-(\mathrm{wt}(u)+\mathrm{wt}(v),\mathrm{wt}(w))}(v \otimes u \otimes w) \tag{11}$$
$$=s_{12}\sum_{\gamma',\gamma''\in Q^+} M_{23}^{\gamma'}q^{-(\gamma'',\mathrm{wt}(u))}M_{13}^{\gamma''}q^{-(\mathrm{wt}(u)+\mathrm{wt}(v),\mathrm{wt}(w))}(v \otimes u \otimes w) \tag{12}$$
$$=s_{12}\sum_{\gamma',\gamma''\in Q^+} M_{23}^{\gamma'}q^{-(\mathrm{wt}(u),\mathrm{wt}(w)+\gamma'')}M_{13}^{\gamma''}q^{-(\mathrm{wt}(u),\mathrm{wt}(w))}(v \otimes u \otimes w) \tag{13}$$
$$=s_{12}M_{23}(A_{23})^{-1}M_{13}(A_{13})^{-1}(v \otimes u \otimes w) \tag{14}$$
$$=s_{12}R_{23}R_{13}s_{12}(u \otimes v \otimes w) \tag{15}$$
$$=R_{13}R_{23}(u \otimes v \otimes w) \tag{16}$$

(iii): Follows by a similar calculation to (ii). $\square$

The following lemma essentially says that the inverse of a braiding is still a braiding.

Lemma 2.11. *$R \in U_q(\mathfrak{g}) \widetilde{\otimes} U_q(\mathfrak{g})$ is a universal R-matrix if and only if R_{21}^{-1} is a universal R-matrix.*

Proof. Assume R is a universal R matrix. It suffices to show that the three conditions of Definition 2.1 hold for R_{21}^{-1}.

(i): Fix $u \in U_q(\mathfrak{g})$.

$$R_{21}^{-1}\Delta(u) = \mathrm{Flip}(R^{-1}\Delta^{op}(u)) \tag{17}$$
$$= \mathrm{Flip}(\Delta(u)R^{-1}) \tag{18}$$
$$= \Delta^{op}(u)R_{21}^{-1}. \tag{19}$$

Here Equation (18) follows by rearranging Definition 2.1 part (i) for R.

(iii): Let s_{321} be the permutation $3 \to 2 \to 1 \to 3$.

$$(\Delta \otimes 1)R_{21}^{-1} = s_{321}(1 \otimes \Delta)R^{-1} \tag{20}$$
$$= s_{321}R_{12}^{-1}R_{13}^{-1} \tag{21}$$
$$= R_{31}^{-1}R_{32}^{-1} \tag{22}$$
$$= (R_{21}^{-1})_{13}(R_{21}^{-1})_{23}. \tag{23}$$

Here Equation (21) follows from Definition 2.1 part (iii) for R.

(iii): Let s_{123} be the permutation $1 \to 2 \to 3 \to 1$.

$$(1 \otimes \Delta)R_{21}^{-1} = s_{123}(\Delta \otimes 1)R^{-1} \tag{24}$$
$$= s_{123}R_{23}^{-1}R_{13}^{-1} \tag{25}$$
$$= R_{31}^{-1}R_{21}^{-1} \tag{26}$$
$$= (R_{21}^{-1})_{13}(R_{21}^{-1})_{12}. \tag{27}$$

Here Equation (25) follows from Definition 2.1 part (ii) for R. □

Proof of Proposition 1.1. By Theorem 2.5 and Proposition 2.10, there is a unique universal R-matrix of the form

$$\Big(1 \otimes 1 + \sum_{\substack{\text{positive integral} \\ \text{weights } \beta \text{ (with} \\ \text{multiplicity)}}} Y_\beta \otimes X_\beta\Big)A^{-1}, \tag{28}$$

where each X_β has weight β and Y_β has weight $-\beta$. By Lemma 2.11, there is also a unique universal R-matrix of the form

$$A^{21}\Big(1 \otimes 1 + \sum_{\substack{\text{positive integral} \\ \text{weights } \beta \text{ (with} \\ \text{multiplicity)}}} X_\beta \otimes Y_\beta\Big)^{-1}. \tag{29}$$

Clearly $A^{21} = A$, and, for some elements X'_β of weight β and Y'_β of weight $-\beta$,

$$\Big(1 \otimes 1 + \sum_{\substack{\text{positive integral} \\ \text{weights } \beta \text{ (with} \\ \text{multiplicity)}}} X_\beta \otimes Y_\beta\Big)^{-1} = \Big(1 \otimes 1 + \sum_{\substack{\text{positive integral} \\ \text{weights } \beta \text{ (with} \\ \text{multiplicity)}}} X'_\beta \otimes Y'_\beta\Big). \tag{30}$$

The proposition follows. □

References

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E-mail address: pwtingle@math.berkeley.edu

UC Berkeley, Department of Mathematics, Berkeley, CA