Peter-Weyl bases, preferred deformations, and Schur-Weyl duality

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Outline

Deformations

Definition

$\mathcal{A}$ an algebraic thing over a field $\mathbb{C}$. A (formal) deformation of $\mathcal{A}$ is the same sort of algebraic thing over $\mathbb{C}[[\hbar]]$ which becomes $\mathcal{A}$ when $\hbar$ is set to 0.

- For us, algebraic thing will be a bialgebra, so $\mathcal{A}$ has multiplication μ and comultiplication Δ (and unit and counit I guess).
- The deformed version in $\mathcal{A}_\hbar$ look like
  - $\mu_\hbar(a,b) = \mu_0(a,b) + \hbar\mu_1(a,b) + \hbar^2\mu_2(a,b) + \cdots$
  - $\Delta_\hbar(a) = \Delta_0(a) + \hbar\Delta_1(a) + h^2\Delta_2(a) + \cdots$

  where μ_0, Δ_0 are the original multiplication and comultiplication.

Definition

Two deformations $\mathcal{A}_\hbar$ and $\mathcal{A}'_\hbar$ are equivalent if they are isomorphic (as bialgebras), using an isomorphism which is the identity at $\hbar = 0$ (i.e. becomes the identity on $\mathcal{A}[[\hbar]]/\hbar\mathcal{A}[[\hbar]]$).

Cases of $U_\hbar(\mathfrak{g})$ and $\mathcal{O}_\hbar(G)$

- In fact, we only really work with $U_\hbar(\mathfrak{g})$ and $\mathcal{O}_\hbar(G)$.
- $U_\hbar(\mathfrak{g})$ is known to be a trivial deformation of the algebra structure, meaning it is equivalent to a deformation where multiplication is unchanged.
- Similarly, $\mathcal{O}_\hbar(G)$ is a trivial deformation of the co-algebra structure.
- The question of a preferred deformation/preferred presentation it to realize these in such a way that mult/comult is literally unchanged.
- Kind of equivalently, we want to identify $\mathcal{O}_\hbar(G)$ (as a vector space) with $\mathcal{O}(G)[[\hbar]]$ is such a way that the comultiplication in $\mathcal{O}_\hbar(G)$ is identified with the natural comultiplication for $\mathcal{O}(G)[[\hbar]]$.
- The normal presentations are not preferred, and hard to see how to “fix."

$U_\hbar(\mathfrak{sl}_2)$

Comultiplication given on generators by

$$\Delta E = E \otimes e^{-\hbar H} + 1 \otimes E$$
$$\Delta F = F \otimes 1 + e^{\hbar H} \otimes F$$
$$\Delta H = H \otimes 1 + 1 \otimes H$$

multiplication described by relations like

$$EF - FE = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}}.$$

The $\hbar$ here makes it pretty non-preferred...this is fixed in CP for $\mathfrak{sl}_2$, but only recently by Appel and Gautam in $\mathfrak{sl}_n$, and not in other cases at all.

$\mathcal{O}_\hbar(\mathrm{SL}_2)$

Undeformed

- Algebra is commutative algebra in the entries a, b, c, d of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$
- Coproduct is defined by $\Delta(f)(M, N) = f(MN)$.
- For generators,

$$\Delta\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\otimes a + b\otimes c & a\otimes b + b\otimes d \\ c\otimes a + d\otimes c & c\otimes b + d\otimes d \end{bmatrix}$$

Deformed:

- Commutativity relations become ($q = e^\hbar$)

$$ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc,$$
$$bc = cb, \quad ad - da = (q - q^{-1})bc.$$

- Comult unchanged on generators, but in higher degree is confusing. e.g.

$$\begin{aligned}\Delta(a^2) &= a^2 \otimes a^2 + ab \otimes ac + ba \otimes ca + b^2 \otimes c^2 \\ &= a^2 \otimes a^2 + (1 + q^{-2})ab \otimes bc + b^2 \otimes c^2.\end{aligned}$$

Peter-Weyl

- Usually a statement about harmonic analysis, we need need simple version:

$$\mathcal{O}(G) \simeq \oplus_\lambda \mathrm{End}(V_\lambda)^*$$

  where the isomorphism is as coalgebras.
- Since the coalgebra structure does not deform, as coalgebras,

$$\mathcal{O}_\hbar(G) \simeq \oplus_\lambda \mathrm{End}_{\mathbb{C}[[\hbar]]}(V_\lambda)^*$$

- To get a preferred deformation, we will understand multiplication in this context! First in an undeformed way.
- Given $a \in \mathrm{End}(V_\lambda)^*, b \in \mathrm{End}(V_\mu)^*$, ab should be

$$a \otimes b \in \mathrm{End}(V_\lambda \otimes V_\mu)^* \simeq \mathrm{End}(V_\lambda)^* \otimes \mathrm{End}(V_\mu)^*.$$

- This is a fine element of $\mathcal{O}(G)$ (since G maps to $\mathrm{End}(V_\lambda \otimes V_\mu)$), but not expressed in $\oplus_\lambda \mathrm{End}(V_\lambda)^*$.

In coordinates

- $\mathrm{End}V_\lambda = V_\lambda \otimes V_\lambda^*$, so $(\mathrm{End}V_\lambda)^* = V_\lambda^* \otimes V_\lambda$.
- There is a basis of matrix elements, $Y^* \otimes X$, for X, Y^* in dual bases for V_λ, V_λ^*. This acts on $g \in G$ as $Y^*(g(X))$.
- Need to express the function defined by
$$(Y_1^* \otimes X_1) \otimes (Y_2^* \otimes X_2) \in \mathrm{End}(V_\lambda)^* \otimes \mathrm{End}(V_\mu)^*$$
as a combinations of matrix elements elements of V_ν's. Same as
$$(Y_1^* \otimes Y_2^*) \otimes (X_1 \otimes X_2) \in \mathrm{End}(V_\lambda \otimes V_\mu)^*$$
- Will need to decompose $V_\lambda \otimes V_\mu$ into irreducibles, and express $X_1 \otimes X_2$ as a sum of elements of these. Similarly for $Y_1^* \otimes Y_2^*$.
- For all ν, fix a basis of embeddings $\psi_1^\nu, \ldots, \psi_{c^\nu}^\nu$ of V_ν in $V_\lambda \otimes V_\mu$. Then,
$$X_1 \otimes X_2 = \begin{pmatrix} \lambda & \mu & \nu \\ X_1 & X_2 & X_3 \end{pmatrix}_k \psi_k(X_3),$$
where these are the “3j" symbols.
- Similarly, there are dual 3j symbols.

Structure constants

If $f = Y_1^* \otimes X_1$, $g = Y_2^* \otimes X_2$,

$$fg = \sum_{\substack{\nu \in P_+ \\ X_3, Y_3 \in B_\nu}} \left[\sum_{1 \leq k \leq c_{\lambda,\mu}^{\nu}} \overline{\begin{pmatrix} \lambda & \mu & \nu \\ Y_1^* & Y_2^* & Y_3^* \end{pmatrix}_k} \begin{pmatrix} \lambda & \mu & \nu \\ X_1 & X_2 & X_3 \end{pmatrix}_k \right] Y_3^* \otimes X_3.$$

Deforming

- We think of V_λ as being a representation of $U(\mathfrak{g})$, not G, then deform to rep of $U_\hbar(\mathfrak{g})$,
- 3j symbols can be defined just fine over $\mathbb{C}[[\hbar]]$, and this all deforms.
- What you need is:
  - A basis for each V_λ over $\mathbb{C}[[\hbar]]$, which specializes to a basis at $\hbar = 0$.
  - A basis for each space of embeddings $V_\nu \hookrightarrow V_\lambda \otimes V_\mu$ over $\mathbb{C}[[\hbar]]$, which specializes to a basis at $\hbar = 0$.
- The set of matrix elements $Y^* \otimes X$ for $X \in B_\lambda, Y^* \in B^*_\lambda$ is a basis for $\mathcal{O}(G)$, and under deformation the coproduct is unchanged. It is preferred!
- Difficulty of multiplication includes calculating a lot of quantum 3j symbols...ok, so there is a cost.

For those who like Schur-Weyl duality

- If the category of representations is generated by a single nice V, there is another natural realization of $\mathcal{O}$: all functions can be realized in
$$\oplus_n \mathrm{End}(V^{\otimes n})^*$$
- Actually this is too big, the same functions are counted many times. But for (polynomial representations of) GL_k, we have something nice:
$$V^{\otimes n} = \oplus_\lambda V_\lambda \otimes W_\lambda,$$
where W_λ range over certain irreps of S_n. Then
$$\mathcal{O}(M_{k\times k}) = \oplus_n (\mathrm{End}_{S_n}(V^{\otimes n}))^*$$
- The construction works in this context, and recovers the preferred deformation of $M_{k\times k}$ (and with extra work $\mathrm{GL}_k, \mathrm{SL}_k$) studied by Giaquinto-Gerstenhaber, Giaquinto, Schack around 1992, which was developed in a very Schure-Weyl dual way.
- Get a slightly different basis, essentially rescaled by $\dim W_\lambda$, which slightly changes structure constants.

Thanks for listening!!!!!!!!