QUIVER VARIETIES AND ROOT MULTIPLICITIES IN RANK 3

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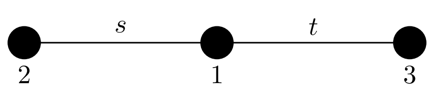
Abstract. Building on our recent work in rank two, we use quiver varieties to give a combinatorial upper bound on dimensions of certain imaginary root spaces for symmetric rank 3 Kac-Moody algebras. We describe an explicit method for extracting combinatorics when the Dynkin diagram is bipartite, so two of the nodes are not connected. As in rank two we believe these bounds are quite tight, and we give computational evidence to this effect, although there is more error in rank 3 than in rank 2.

1. Introduction

In [Tin21], we developed a general method for studying root multiplicities of symmetrizable Kac-Moody algebras using quiver varieties. This gives a framework for finding combinatorial upper bounds on the multiplicities, and we conjectured that the resulting bounds, at least in some cases, are quite tight. While the construction is general, we only translated it to combinatorics in rank two. There the upper bound consisted of the number of Dyck paths satisfying various conditions.

Here we develop explicit combinatorics using the same method in certain rank three cases. Specifically, we consider a Kac-Moody algebra $\mathfrak{g}$ with Dynkin diagram



meaning there are s edges on the left and t edges on the right. Our main result is the following:

Theorem 1.1. *For any imaginary root $\beta = a\alpha_1 + b\alpha_2 + c\alpha_3$ with $gcd(a,b,c) = 1$, the root multiplicity m_β is bounded above by the number of words of the form*

$$1^{a_1}2^{b_1}3^{c_1}1^{a_2}2^{b_2}3^{c_2}\cdots, \tag{1.2}$$

where the total number of 1s is a, the number of 2s is b, the number of 3s is c, and,

1. *For each i, $a_i \neq 0$ and b_i or c_i is also non-zero.*
2. *Draw a path in the plane by drawing each 1 as a vertical line of length 1, each 2 as a horizontal line of length s, and each 3 as a horizontal line of length t. The result is a rational Dyck path. That is, a straight line connecting the beginning and end of the path stays weakly below the path.*
3. *If a prefix $1^{a_1}2^{b_1}3^{c_1}\cdots 1^{a_k}2^{b_k}3^{c_k}$ corresponds to a point where the Dyck path touches the diagonal, then $\dfrac{b_1+\cdots+b_k}{c_1+\cdots+c_k} \geq \dfrac{b}{c}$. Equivalently, if you deform the path to have each 2 correspond to an edge of length $s-\epsilon$, the resulting path still stays above its diagonal.*
4. *For each $i \geq 1$, $\dfrac{b_i}{a_i} \leq s$ and $\dfrac{c_i}{a_i} \leq t$.*

For each $i \geq 1$, *let* $n_{bi} = \min\{b_i, sa_{i+1} - b_{i+1}\}$ *and* $n_{ci} = \min\{c_i, ta_{i+1} - c_{i+1}\}$.

(5) $a_{i+1} \leq sn_{bi} + tn_c - \max\{s^{-1}n_b, t^{-1}n_{ci}\}$.

(6) $\dfrac{a_{i+1}}{sn_{bi} + tn_{ci}} \leq \dfrac{1}{2} + \dfrac{\sqrt{(s^2+t^2)^2 - 4(s^2+t^2)}}{2(s^2+t^2)}$.

Similar methods should work for any symmetric Kac-Moody algebra whose Dynkin diagram is bipartite, with the combinatorics being most similar when the Dynkin diagram is a star. The method from [Tin21] is even more general, but as the diagram gets more complex it is harder to describe things combinatorially.

Unlike in the rank two case from [Tin21], we don't believe that Theorem 1.1 gives the complete list of "local" conditions on the path, so the upper bound will diverge exponentially from the correct answer. Nonetheless, we give evidence that the bounds are meaningful. We also discuss ways to tighten the bounds, although we don't believe that giving the full list of conditions is feasible.

Root multiplicities have been studied quite extensively, see [CFL14] for a survey. They can be calculated exactly, see [BM79, Pet83], although the formulae are complicated. Special cases have been further investigated and combinatorialized in [FF83, KLL17, KM95]. But open questions remain. See e.g. [CFL14, Open Problems 2 and 3] and Frenkel's conjectural upper bound for hyperbolic cases [Fre85]. We hope our work sheds some light on the situation, particularly on asymptotics.

2. Background

2.1. Kac-Moody algebras and $B(-\infty)$. We consider the Kac-Moody algebra $\mathfrak{g}$ associated to a symmetric Cartan matrix $A = (a_{ij})$ with index set I, see [Kac90]. The Lie algebra $\mathfrak{g}$ is graded by the root lattice Q, which is the $\mathbb{Z}$-span of the simple roots α_i for $i \in I$. By definition a non-zero $\beta \in Q$ is a root if $\dim \mathfrak{g}_\beta \neq 0$, in which case $m_\beta := \dim \mathfrak{g}_\beta$ is called the root multiplicity. All roots are either positive, meaning they are $\mathbb{Z}_{\geq 0}$ linear combinations of the α_i, or negative, meaning the negatives of these. Let Δ denote the set of roots and Δ_+ the positive roots.

There is an inner product on Q defined by, for simple roots α_i, α_j, $\langle \alpha_i, \alpha_j \rangle = a_{ij}$. All roots β have the property that either $\langle \beta, \beta \rangle = 2$, in which case β is called a real root, or $\langle \beta, \beta \rangle \leq 0$, in which case β is called an imaginary root. The refection corresponding to each α_i acts on weight space by

$$s_i(\nu) = \nu - \langle \alpha_i, \nu \rangle \alpha_i.$$

This has various important properties, including:

- It preserves the inner product.
- It preserves root multiplicities.
- It preserves the set of positive imaginary roots, meaning it acts as a permutation on this set.

2.2. The crystal $B(-\infty)$. We don't work directly with $\mathfrak{g}$ here, but instead with the crystal $B(-\infty)$. This is a combinatorial object related most closely not to $\mathfrak{g}$, but to its universal enveloping algebra $U(\mathfrak{g})$. As a vector space,

$$U(\mathfrak{g}) = U^-(\mathfrak{g}) \otimes U^0(\mathfrak{g}) \otimes U^+(\mathfrak{g}),$$

where U^-, U^0, U^+ are the subalgebras generated by the negative root spaces, the Cartan subalgebra, and the positive root spaces, respectively. The graded dimension of U^+ is

$$\dim U^+ = \prod_{\beta \in \Delta_+} \left(\frac{1}{1-e^\beta} \right)^{m_\beta}.$$

That is, the dimension of the γ-weight space of $U^+(\mathfrak{g})$ is the number of ways to write γ as a sum of positive roots, taking into account multiplicities. This is also called the Kostant partition function of γ.

The crystal $B(-\infty)$ is a set along with operators $e_i, f_i \colon B(-\infty) \to B(-\infty) \cup \{0\}$ for each $i \in I$, which satisfy various axioms. There is a weight function wt : $B(-\infty) \to Q$, and the number of elements of a given weight γ is the dimension of the γ weight space in $U^+(\mathfrak{g})$. See [Kas95] or [HK02] (which consider $B(\infty)$, the Cartan involution of $B(-\infty)$). Here we use the geometric realization of $B(-\infty)$ from [KS97], which is explained below.

2.3. **Quiver varieties.** Fix a graph G with vertex set I and edge set E. Let A be the set of arrows, so there are two arrows for each edge $e \in E$, one pointing in each direction. For each arrow a, let $s(a)$ be the source and $t(a)$ be the target, meaning a points from $s(a)$ to $t(a)$.

Definition 2.1. The path algebra $\mathbb{C}[G]$ is the $\mathbb{C}$-algebra with basis consisting of all paths in G (sequences of arrows $a_k \cdots a_1$ with $t(a_i) = s(a_{i+1})$), along with the lazy paths e_i at each vertex) and with multiplication given by

$$(b_k \cdots b_1)(a_j \cdots a_1) = \begin{cases} b_k \cdots b_1 a_j \cdots a_1 & t(a_j) = s(b_1) \\ 0 & \text{otherwise.} \end{cases}$$

Choose a subset Ω of A where each edge appears in exactly one direction (sometimes called an orientation of the graph) and define $\epsilon(a) = 1$ if $a \in \Omega$ and -1 otherwise. For $a \in A$, let $\bar{a}$ denote the reverse arrow.

Definition 2.2. The preprojective algebra Λ is the quotient of $\mathbb{C}[G]$ by the ideal generated by

$$\epsilon = \sum_{a \in A} \epsilon(a) \bar{a} a.$$

Definition 2.3. For any I-graded vector space $V = \oplus_I V_i$, let $\Lambda(V)$ be the variety of actions of Λ on V where the lazy path at i acts as projection onto V_i, and which are nilpotent in the sense that all paths of length at least $\dim V$ act as 0.

A representation of $\mathbb{C}[G]$ is determined by a homomorphism for each arrow, so can be described as

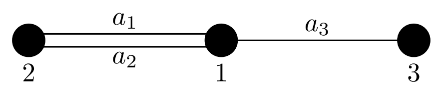
$$(x_a)_{a \in A} \in \oplus_A \mathrm{Hom}(V_{s(a)}, V_{t(a)}).$$

$\Lambda(V)$ is a sub-variety of this space. Up to isomorphism it depends only on the dimension vector $\boldsymbol{v} = (v_i)_{i \in I}$, where $v_i = \dim V_i$.

Associate to G a symmetric Cartan matrix whose index set is the set of vertices, the diagonal entries are 2, and other entires are defined by setting $-a_{ij}$ to be the number of edges connecting i and j. We identify a dimension vector $\boldsymbol{v}$ with $\gamma = \sum_i v_i \alpha_i \in P$, and sometimes denote $\Lambda(V)$ by $\Lambda(\gamma)$. Denote the set of irreducible components of $\Lambda(\gamma)$ by $\boldsymbol{Irr}\Lambda(\gamma)$. The following is due to Kashiwara and Saito [KS97], so also [NT18, Proposition 3.14].

Theorem 2.4. *The crystal $B(-\infty)$ is naturally indexed by $\coprod \boldsymbol{I}rr\Lambda(\gamma)$. The operation f_i^{max} which applies the crystal operator f_i as many times as possible acts on $X \in \boldsymbol{I}rr\Lambda(\boldsymbol{v})$ as follows: Fix $T \in X$. Let $Soc_i(T)$ be the intersection of the socle of T with V_i and set $\gamma' = \gamma - \dim Soc_i(T)\alpha_i$. Generically $T/Soc_i(T)$ is isomorphic to a point in a unique $Y \in \boldsymbol{I}rr\Lambda(\gamma')$, and $f_i^{max}X = Y$.* ■

Example 2.5. The case $s = 2, t = 1$ is particularly important here. That is, the Dynkin diagram



with two edges on the left, and one on the right. Orient all the edges to point away from the center node, and call the oriented arrows a_1, a_2, a_3. A representation of $\mathbb{C}[G]$ on $V_0 \oplus V_1 \oplus V_2$ consists of 6 maps:

$$x_{a_1}, x_{a_2} : V_1 \to V_2, \quad x_{a_3} : V_1 \to V_3, \quad x_{\bar{a}_1}, x_{\bar{a}_2} : V_2 \to V_1, \quad x_{\bar{a}_3} : V_3 \to V_1.$$

$\Lambda(v)$ is the sub-variety cut out by the condition that all paths of length greater than $\dim V_0 + \dim V_1 + \dim V_2$ act as 0, and the equations

$$\bar{a}_1 a_1 + \bar{a}_2 a_2 + \bar{a}_3 a_3 = 0, \quad a_1 \bar{a}_1 + a_2 \bar{a}_2 = 0, \quad a_3 \bar{a}_3 = 0,$$

where these are equations in $\mathrm{End}(V_1), \mathrm{End}(V_2), \mathrm{End}(V_3)$ respectively.

2.4. **Stability conditions.** We loosely follow [BKT14], drawing on notation from [TW16] as well. See also [Tin21].

Define a **charge** c to be a linear function $c : P \to \mathbb{C}$ such that $c(\alpha_i)$ is in the upper half plane for all simple roots α_i. For a fixed charge c, each representation V of Λ has a unique filtration

$$\emptyset = V_0 \subset V_1 \subset \cdots \subset V_k = V$$

where the sub-quotients $Q_i = V_i/V_{i-1}$ satisfy

(HN1) Q_i has no submodule S with $\arg(c(\dim S)) < \arg(c(\dim Q_i))$,
(HN2) $\arg(c(\dim Q_1)) < \arg(c(\dim Q_2)) < \cdots < \arg(c(\dim Q_k))$.

Here $\arg(z)$ is the angle formed by $z, 0$ and 1 in the complex plane. We call this the HN filtration since it is a special case of a Harder-Narasimhan filtration as in e.g. [Rud97].

Fix a stability condition c so that, for any root β, if $\arg c(\alpha) = \arg c(\beta)$ then β and α are parallel. The following can be extracted from [BKT14], see [Tin21, Theorem 3.3] for the exact statement.

Theorem 2.6. *For any $\gamma \in Q_+$, the number of stable irreducible components of $\Lambda(\gamma)$ is the sum over all ways of writing $\gamma = \beta_1 + \cdots + \beta_n$ as a sum of parallel roots β_k of the product $m_{\beta_1} \cdots m_{\beta_n}$ of the corresponding root multiplicities. In particular, if γ is not parallel to any smaller weight, it is exactly m_γ.*

2.5. **String data.** The following was first studied by Kashiwara [Kas95, §8.2] and was further developed by Littelmann [Lit98]. Choose a sequence $i_1, i_2, i_3 \ldots$ of nodes in the Dynkin diagram with each appearing infinitely many times. The **string data** $(a_1, a_2, \ldots)$ of $b \in B(-\infty)$ is

$$a_1 = \max\{n : f_{i_1}^n b \neq 0\},$$
$$a_2 = \max\{n : f_{i_2}^n f_{i_1}^{a_1} b \neq 0\},$$

and so on. Record this as a word in the letters I consisting of a_1 i_1's, followed by a_2 i_2's, and so on. Sometimes we write this as

$$i_1^{a_1} i_2^{a_2} \cdots i_k^{a_k}.$$

The string data uniquely determines an element $b \in B(-\infty)$.

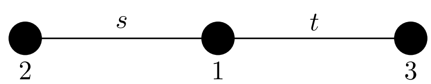
Indexing $B(-\infty)$ by $\sqcup \boldsymbol{I}rr\Lambda(\boldsymbol{v})$, Theorem 2.4 shows that the string data of $X \in \boldsymbol{I}rr\Lambda(\boldsymbol{v})$ gives the dimensions of the graded socle filtration of a generic $T \in X$:

$$\begin{aligned} a_1 &= \dim \operatorname{Hom}(\mathbb{C}_{i_1}, T), \\ a_2 &= \dim \operatorname{Hom}(\mathbb{C}_{i_2}, T/i_1\text{socle}), \end{aligned}$$

and so on, where $\mathbb{C}_i$ is the one dimensional simple module in degree i.

3. Rank three and proof of Theorem 1.1

As in the introduction, consider the case where the Dynkin diagram is



The corresponding Cartan matrix is

$$\begin{pmatrix} 2 & -s & -t \\ -s & 2 & 0 \\ -t & 0 & 2 \end{pmatrix}.$$

Consider the charge defined by

$$c(\alpha_1) = -1 + i, c(\alpha_2) = s + si, c(\alpha_3) = t - \epsilon + ti$$

where ϵ is infinitesimal. Then $\arg(c(a_1\alpha_1 + b_1\alpha_2 + c_1\alpha_3)) < \arg(c(a_2\alpha_1 + b_2\alpha_2 + c_2\alpha_3)$ exactly if

$$\begin{cases} \frac{sb_1+tc_1}{a_1} > \frac{sb_2+tc_2}{a_2}, \text{ or} \\ \frac{sb_1+tc_1}{a_1} = \frac{sb_2+tc_2}{a_2} \text{ and } \frac{b_1}{c_1} < \frac{b_2}{c_2}. \end{cases}$$

If $\beta = a\alpha_b + q\alpha_2 + c\alpha_3$ and $\gcd(a, b, c) = 1$, then Theorem 2.6 shows that m_β is the number of stable irreducible components of $\Lambda(\beta)$.

Remark 3.1. We could use any charge. This one is convenient because for technical reasons it gives slightly simpler conditions. With a different charge, the analogue of Theorem 1.1 requires some extra special cases.

Fix $T \in \Lambda(a\alpha_1 + b\alpha_2 + c\alpha_3)$. Take string data as in §2.5 using the sequence $1, 2, 3, 1, 2, 3, \cdots$. It is convenient to use specialized notation, where the string data is $a_1, b_1, c_1, a_2, b_2, c_2, \cdots$, so that, for T generic in X,

$$a_1 = \dim \operatorname{Hom}(\mathbb{C}_1, T)$$

$$b_1 = \dim \operatorname{Hom}(\mathbb{C}_2, T_1), \text{ where } T_1 = T/1 - \text{socle}$$

$$c_1 = \dim \operatorname{Hom}(\mathbb{C}_3, T_2), \text{ where } T_2 = T_1/2 - \text{socle}$$

$$a_2 = \dim \operatorname{Hom}(\mathbb{C}_1, T_3), \text{ where } T_3 = T_2/3 - \text{socle},$$

and so on. For each word, create a path in $\mathbb{R}^2$ where a 1 corresponds to a step up, a 2 to s steps to the right, and a 3 to t steps to the right.

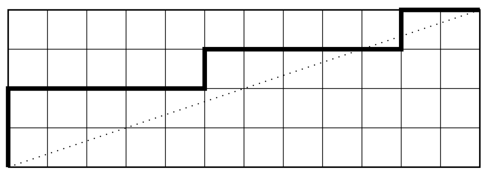
Lemma 3.2. *If $T \in \Lambda(a\alpha_1 + b\alpha_2 + c\alpha_3)$ is a stable module with string data $a_1, b_1, c_1, a_2...$, then this is a rational Dyck path. Furthermore, for any k where the path touches the diagonal, $\frac{b_1+\cdots+b_k}{c_1+\cdots+c_k} \geq \frac{b}{c}$. That is, Theorem 1.1 parts (2) and (3) hold.*

Proof. Each left-prefix of the sequence corresponds to a sub-model of T, and if the conditions do not hold then this gives a submodule that violates stability. ∎

Example 3.3.

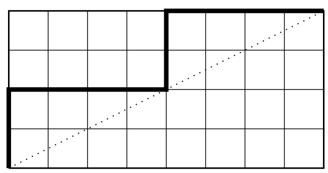
11223122312

becomes the path



which is not a rational Dyck path because it dips below the diagonal at $(10, 3)$.

Example 3.4. Both 112211233 and 112331122 correspond to the Dyck path



But the submodule of 112331122 corresponding to 11233 still fails stability because it touches the diagonal and has a lower ratio of 2s to 3s than the whole word. 112211233 does correspond to a stable component.

We now need a little more notation. If a module M has string data $a_1 b_1 c_1 a_2 b_2...$, then we can choose a decomposition

$$M = A_1 \oplus B_1 \oplus C_1 \oplus \cdots \oplus A_k \oplus B_k \oplus C_k \tag{3.5}$$

which is compatible with the socle filtration in the sense that

- The 1-socle is A_1,
- The 2 socle of M/A_1 is B_1, That is, the natural map from B_1 to the 2-socle of the quotient is an isomorphism.
- The 3 socle of $M/(A_1 \oplus B_1)$ is C_1,
- The 1 socle of $M/(A_1 \oplus B_1 \oplus C_1)$ is A_2,

and so on. Then the representation induces various maps for each i:

$$_1x_2 = \bigoplus_{s(a)=2,t(a)=1} x_a : B_i \to A^s_{i-1}, \quad {}_2x_1 = \bigoplus_{s(a)=1,t(a)=2} x_a : A_i \to B^s_{i-1},$$

$$_1x_3 = \bigoplus_{s(a)=3,t(a)=1} x_a : C_i \to A^s_{i-1}, \quad {}_3x_1 = \bigoplus_{s(a)=1,t(a)=3} x_a : A_i \to C^t_{i-1},$$

$$_1\bar{x}_2 = \sum_{s(a)=2,t(a)=1} x_a : B^s_i \to A_{i-1}, \quad {}_2\bar{x}_1 = \sum_{s(a)=1,t(a)=2} x_a : A^s_i \to B_{i-1},$$

$$_1\bar{x}_3 = \sum_{s(a)=3,t(a)=1} x_a : C^s_i \to A_{i-1}, \quad {}_3\bar{x}_1 = \sum_{s(a)=1,t(a)=3} x_a : A^t_i \to C_{i-1}.$$

These are actually families of maps depending on i, and we put a superscript of i on the maps if we need to specify which we mean, so a superscript of i means that the domain of the map is A_i, B_i or C_i. But in most contexts that is clear and we suppress the notation. The preprojective relations imply that

$$ {}_2\bar{x}_1 \circ {}_1x_2 = 0, \quad {}_1\bar{x}_2 \circ {}_2x_1 + {}_1\bar{x}_3 \circ {}_3x_1 = 0, \quad {}_3\bar{x}_1 \circ {}_1x_3 = 0, $$

and this holds with any superscript i.

Below we repeatedly use the following facts that a generic module in a stable irreducible component is stable and so, if one can show that there are no stable modules with a given string data, there can be no stable irreducible component with that data.

fact that any sub-word of a valid string data must be a valid string data.

Lemma 3.6. *There is no stable Λ-module whose string data has $b_i > sa_i$ or $c_i > ta_i$ for any i. In particular, Theorem 1.1(4) holds.*

Proof of Lemma 3.6. Assume a module M has this string data, and $b_i > sa_i$. Then by dimension count ${}_1x_2 : B_i \to A_i^{\oplus s}$ must have a kernel. If $i > 1$ this contradicts the definition of the socle filtration, and if $i = 1$ it contradicts stability.

It follows that Theorem 1.1(4) holds for any stable component since it holds for any stable module. ■

Lemma 3.7. *Assume a Λ module has string data $a_1, b_1, c_1, \cdots b_k, c_k$ and is stable. Fix i, and let $n_{bi} = \min\{b_i, sa_{i+1} - b_{i+1}\}, n_{ci} = \min\{c_i, ta_{i+1} - c_{i+1}\}$. Then*

$$ a_{i+1} \leq sn_b + tn_c - \max\{s^{-1}n_b, t^{-1}n_c\}. $$

In particular, Theorem 1.1(5) holds.

Proof. By the definition of the socle filtration, ${}_1x_2 : B_{i+1} \to A_{i+1}^{\oplus s}$ is injective. Also, by the preprojective relations,

$$ {}_2\bar{x}_1^i \circ {}_1x_2^{i+1} : B_{i+1} \to B_i $$

is the zero map. Hence

$$ \dim \operatorname{im} {}_2\bar{x}_1 \leq \dim A_{i+1}^{\oplus s} - \dim B_{i+1} = sa_{i+1} - b_{i+1}. $$

Certainly

$$ \dim \operatorname{im} {}_2\bar{x}_1 \leq \dim B_i = b_i. $$

Putting this together,

$$ \dim \operatorname{im} {}_2\bar{x}_1^i \leq n_{bi}. $$

Similarly, $\dim \operatorname{im} {}_2\bar{x}_3^i \leq n_{ci}$.

Let $I_b = \operatorname{im} {}_2\bar{x}_1 \subset B_i, I_c = \operatorname{im} {}_3\bar{x}_1 \subset C_i$, Again using the definition of a socle filtration,

$$ {}_2x_1^i \oplus {}_3x_1^i : A_{i+1} \to I_b^{\oplus s} \oplus I_c^{\oplus t} $$

is injective, so the dimension of its image is a_{i+1}.

Now, ${}_1x_2|_{I_b}$ is injective, so its image has dimension n_{bi}. Recalling that

$$ {}_1x_2|_{I_b} = \bigoplus_{a:s(a)=2,t(a)=1} x_a, $$

at least one of these a must have $\dim \operatorname{im} x_a|_{I_b} > s^{-1}n_{bi}$. This in turn implies that $\dim \operatorname{im} {}_1\bar{x}_2^i|_{I_b^s} \geq s^{-1}n_{bi}$. Similarly, $\dim \operatorname{im} {}_1\bar{x}_3^i|_{I_c^t} \geq t^{-1}n_{ci}$. So,

$$ \dim \operatorname{im} {}_1\bar{x}_2^i + {}_1\bar{x}_3^i|_{I_b^s + I_c^t} \geq \max\{s^{-1}n_{bi} + t^{-1}n_{ci}\}. $$

Using the preprojective relation again,

$$({}_1\bar{x}^i_2 + {}_1\bar{x}^i_3) \circ ({}_2x^i_1 \oplus {}_3x^i_1) = 0.$$

Hence

$$a_{i+1} = \dim \operatorname{im}({}_2x^i_1 \oplus {}_3x^i_1) \le \dim \ker({}_1\bar{x}^i_2 + {}_1\bar{x}^i_3)|_{I^s_b + I^t_c} \le sn_{bi} + tn_{ci} - \max\{s^{-1}n_{bi} + t^{-1}n_{ci}\}.$$

Again, this implies the statement on components. ∎

Lemma 3.8. *Fix $\gamma = a\alpha_1 + b\alpha_2 + c\alpha_3$ with $a, b, c \ge 0$.*

- *Assume $\frac{a}{sb+tc} < \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2 - 4(s^2+t^2)}}{2(s^2+t^2)}$, and that $s_2s_3\beta = a\alpha_1 + b'\alpha_2 + c'\alpha_3$ has $b', c' \ge 0$. Then $\frac{a}{sb'+tc'} > \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$.*
- *Assume $\frac{a}{sb+tc} > \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$ and that $s_1\beta = a'\alpha_1 + b\alpha_2 + c\alpha_3$ has $a' \ge 0$. Then $\frac{a}{sb'+tc'} < \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$.*

Proof. Consider the first case. Then

$$s_2s_3\beta = a\alpha_1 + (sa-b)\alpha_2 + (ta-c)\alpha_3.$$

$$\begin{aligned}
\frac{a}{s(sa-b)+t(ta-c)} &= \frac{a}{(s^2+t^2)a - (sb+tc)} \\
&= \frac{1}{s^2+t^2 - \frac{sb+tc}{a}} \\
&\ge \frac{1}{s^2+t^2 - \frac{1}{\frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}}} \\
&= \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)},
\end{aligned}$$

where the last step requires some simplification. The second case is similar. ∎

Lemma 3.9. *For any imaginary root $a\alpha_1 + b\alpha_2 + c\alpha_3$,*

$$\frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)} < \frac{a}{sb+tc} < \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}.$$

Proof. Proceed by induction on $a+sb+tc$, showing that no β violating the condition is imaginary. The base case is trivial, since if $a+sb+tc = 1$, there are no imaginary roots. So, fix β with $a+sb+tc > 1$. Then $s_2s_3\beta = a\alpha_1 + b'\alpha_2 + c'\alpha_3$ has $b', c' \ge 0$, since reflection preserves the set of positive imaginary roots. If $\frac{a}{sb+tc} < \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$, then, by Lemma 3.8, $s_2s_3\beta = a\alpha_1 + b'\alpha_2 + c'\alpha_3$ satisfies

$$\frac{a}{sb'+tc'} > \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}.$$

This certainly also implies $a + sb' + tc' < a + sb + tc$. So by induction $s_2s_3\beta$ is not an imaginary root, and hence neither is β. The other case is similar. ∎

Lemma 3.10. *Fix a module M, with the decomposition above.*

- *Assume that, for some i, $A_i \oplus B_i \oplus C_i$ has a submodule of dimension a', b', c' with $\frac{a'}{sb'+tc'} < \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$. Then $A_i \oplus B_{i-1} \oplus C_{i-1}$ has a submodule of dimension a', b'', c'' with $\frac{a'}{sb''+tc''} > \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$.*
- *Assume that, for some i, $A_i \oplus B_{i-1} \oplus C_{i-1}$ has a submodule of dimension a', b', c' with $\frac{a'}{sb'+tc'} > \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$. Then $A_{i-1} \oplus B_{i-1} \oplus C_{i-1}$ has a submodule of dimension a'', b', c' with $\frac{a''}{sb'+tc'} < \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$.*

Proof. Consider the first case. By the preprojective relation

$$ {}_2\bar{x}_1 \circ {}_1x_2 = 0. $$

The map ${}_1x_2$ is injective, so $\dim \operatorname{im} {}_2\bar{x}_1 \leq sa' - b'$. Similarly, $\dim \operatorname{im} {}_3\bar{x}_1 \leq ta' - c'$. This implies the existance of a submodule of $A_i \oplus B_{i-1} \oplus C_{i-1}$ of dimension a', b'', c'' with

$$ \frac{a'}{sb''+tc''} > \frac{a'}{s(sa'-b')+t(ta'-c')}. $$

This last fraction is the ratio for $s_2 s_3(a'\alpha_1 + b'\alpha_2 + c'\alpha_3)$, so is larger than $\frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}$ by Lemma 3.8.

The other case is similar. ∎

Lemma 3.11. *Fix a stable module. For any i and any submodule $A'_{i+1} \oplus B'_i \oplus C'_i$ of $A_{i+1} \oplus B_i \oplus C_i$, with $\dim A'_{i+1} = a'_{i+1}, \dim B'_i = b'_i, \dim C'_i = c'_i$,*

$$ \frac{a_{i+1}}{sn_{bi}+tn_{ci}} \leq \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}. \tag{3.12} $$

In particular,

$$ \frac{a_{i+1}}{sn_{bi}+tn_{ci}} \leq \frac{1}{2} + \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}, \tag{3.13} $$

so Theorem 1.1(6) holds.

Proof. Assume there is a sub-module where (3.12) fails for some i. If $i = 1$, then, by Lemma 3.10, $A_1 \oplus B_1 \oplus C_1$ has a submodule $A'_1 \oplus B'_1 \oplus C'_1$ of dimension (a'_1, b'_1, c'_1) with

$$ \frac{a'_1}{sb'_1+tc'_1} < \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}. $$

Since $a\alpha_1 + b\alpha_2 + c\alpha_3$ is imaginary, Lemma 3.9 shows that

$$ \frac{a}{sb+tc} > \frac{1}{2} - \frac{\sqrt{(s^2+t^2)^2-4(s^2+t^2)}}{2(s^2+t^2)}. $$

Taking reciprocals we see that $A'_1 \oplus B'_1 \oplus C'_1$ violates stability.

If $i > 1$ then applying Lemma 3.10 twice gives a sub-quotient for a lower i which violates the same condition, and one proceeds by induction.

To see (3.13), notice that the preprojective relation implies that the image of ${}_2x_1 \oplus {}_3x_1; A_{i+1} \to B_i \oplus C_i$ has dimension at most (n_{bi}, n_{ci}), so if this equation is violated it immediately gives a sub-quotient violating (3.12) ∎

This completes the proof of Theorem 1.1.

4. Examples

4.1. **The case $s = 2, t = 1$.** In this case many root multiplicities can be found in [Kac90, Chapter 11]. To estimate the multiplicity of $\beta = a\alpha_1 + b\alpha_2 + c\alpha_3$, Theorem 1.1 says we should count words in 1,2,3 such that

- The resulting path is a Dyck path.
- If a prefix $1^{a_1}2^{b_1}3^{c_1}\cdots 1^{a_k}2^{b_k}3^{c_k}$ corresponds to a point where the Dyck path touches the diagonal, then $\frac{b_1+\cdots+b_k}{c_1+\cdots+c_k} > \frac{b}{c}$.
- $\frac{b_i}{a_i} \leq 2$, $\frac{c_i}{a_i} \leq 1$

Let $n_{bi} = \min\{b_i, 2a_{i+1} - b_{i+1}\}$, $n_{ci} = \min\{c_i, a_{i+1} - c_{i+1}\}$,

- $a_{i+1} \leq 2n_{bi} + n_{ci} - \max\{\frac{n_{bi}}{2}, n_{ci}\}$.
- $\frac{a_{i+1}}{2n_{bi} + n_{ci}} \leq \frac{1}{2} + \frac{\sqrt{5}}{10}$.

Example 4.1. For $\beta = 3\alpha_0 + 4\alpha_1 + 2\alpha_2$, six paths satisfy these conditions:
112312123
112123123
111223123
112211233
112311223
111122233
However, the root multiplicity is 5. The word which does not correspond to a valid stable component is
112311223.
The reason is that the sub-quotient Q corresponding to the sub-string 2311 (shown in red) has the property that the $\mathbb{C}_2$ and $\mathbb{C}_3$ together have only the freedom to map to a single copy of $\mathbb{C}_1$ so it implies the existence of a submodule with socle filtration 123, which violates stability. This path will be elliminated by the refined conditions in the next section.

Example 4.2. Consider $\beta = 33\alpha_1 + 20\alpha_2 + 6\alpha_3$. String data with $a_i = 33, b_i = 20, c_i = 6$ satisfies all of our conditions. But,

$$s_2s_3s_1s_2s_3s_1\beta = 6\alpha_1 + 6\alpha_2 - \alpha_3,$$

so this should not be allowed as the dimension of a submodule of $B_{i-1} \oplus C_{i-1} \oplus A_i$. So, for instance,

$$1^{10}2^{10}3^{10}1^{15}2^{20}3^{6}1^{33}$$

does not correspond to the socle filtration of any module but appears in our count. As with all such cases, we could add a condition to rule this out, but one could just find a more complex example. But, in any case, our ratio condition fails to capture all cases where

$$\cdots s_2s_3s_1s_2s_3s_1s_2s_3s_1\beta$$

results in a negative coefficient. This β has $|\beta|^2 = -110$, but does not correspond to an imaginary root. So, this issue here is related to the existence of elements of the positive root lattice of negative norm which are not imaginary roots.

This particular example would be ruled out by Theorem 5.1 below, although there are others that would not.

4.2. **The case $s = t = 2$.** . To estimate the multiplicity of $\beta = a\alpha_1 + b\alpha_2 + c\alpha_3$, Theorem 1.1 now says we should count words in 1,2,3 such that

- The resulting path is a Dyck path.
- If a prefix $1^{a_1}2^{b_1}3^{c_1}\cdots 1^{a_k}2^{b_k}3^{c_k}$ corresponds to a point where the Dyck path touches the diagonal, then $\frac{b_1+\cdots+b_k}{c_1+\cdots+c_k} > \frac{b}{c}$.
- $\frac{b_i}{a_i}, \frac{c_i}{a_i} \leq 2$

Let $n_{bi} = \min\{b_i, 2a_{i+1} - b_{i+1}\}$, $n_{ci} = \min\{c_i, 2a_{i+1} - c_{i+1}\}$,

- $a_{i+1} \leq 2n_{bi} + 2n_{ci} - \max\{\frac{n_{bi}}{2}, \frac{n_{ci}}{2}\}$.
- $\frac{a_{i+1}}{2n_{bi}+2n_{ci}} \leq \frac{1}{2} + \frac{\sqrt{2}}{4}$.

Example 4.3. A word of the form

$$...112211221112...$$

does not immediately fail any of the conditions. But, the segment colored red corresponds to the sub-quotient Q with socle data 22111. By the usual dimension counts, this implies a submodule with data

$$...1122122.$$

Now the 1 in the sub-quotient Q' indicated in red has no freedom to map to 2s below, so must be part of the socle. This contradicts the fact that this is a socle filtration, so no words of this form correspond to stable components. Note that if the word was of the form

$$...112311221112...$$

instead there would have been no problem.

We can of course write down conditions to rule out Example 4.3. But then one can just find worse examples. For instance:

Example 4.4. Consider now string data of the form

$$....1^?2^71^72^71^72^71^72^71^82^?...$$

Then the 2^71^8 at the right implies a 1^62^7 then a 2^51^6 and so on down to a 21^2 subquotient, and then the two has no freedom to map to any 1s, causing a contradiction. But

$$....1^?2^731^72^71^72^71^72^71^82^?...$$

would not have had this problem. So we need to consider many steps in the path to find the problem.

Example 4.5. Comparing to [Tin21], it seems like the condition on the ratio should be saying that, for any i, either a_{i+1} is small compared to n_b, n_c, or $\nu = a_{i+1}\alpha_1 + n_b\alpha_2 + n_c\alpha_3$ pairs negatively with itself. However, this is not quite right. For example, take $\nu = 19\alpha_1 + 11\alpha_2 + \alpha_3$. Then $||\nu||^2 = 27$. But, the filtration of the maximal module with this head is

$$\cdots 2^{29}3^{21}1^{15}2^13^91^52^{11}3^11^{19},$$

which does not imply a problem with the socle filtration. What we really need to capture is when the sequence $\cdots s_2s_3s_1s_2s_3s_1s_2s_3s_1$ acts on $a_{i+1}\alpha_1 + n_b\alpha_2 + n_c\alpha_3$ to eventually give a negative coefficient. This will never happen for imaginary roots, but may or may not happen for real roots, or for vectors which are not roots. Theorem 1.1(6) rules out certain $a_{i+1}\alpha_1 + n_b\alpha_2 + n_c\alpha_3$ where the coefficient of α_1 must decrease indefinitely each time one applies $s_2s_3s_1$ until one of the coefficients becomes negative, but there are other cases.

5. Refined conditions

In [Tin21], two types of conditions were given: [Tin21, Theorem 4.3] gave local conditions that must be satisfied anywhere on the Dyck path of a stable component, and [Tin21, Theorem 4.9] gave extra conditions that must be satisfied near the beginning of the path, and at places where it is near the diagonal. The previous sections here gave conditions analogous to [Tin21, Theorem 4.3]. We now give some conditions analogous to [Tin21, Theorem 4.9].

Theorem 5.1. *Fix $\beta = a\alpha_1 + b\alpha_2 + c\alpha_3$ and a path $1^{a_1}2^{b_1}3^{c_1}\cdots 1^{a_k}2^{b_k}3^{c_k}$ with $a_1 + \cdots + a_k = a, b_1 + \cdots + b_k = b, c_1 + \cdots + c_k = c$. For any $x \leq y < k$, let*

- $n_{aI} = a_{x+1} + a_{x+2} + \cdots + a_{y+1}$
- $n_{bI} = \min(b_x + b_{x+1} + \cdots + b_y, sn_{aI} - (b_{x+1} + b_{x+2} + \cdots + b_{y+1}))$
- $n_{cI} = \min(c_x + c_{x+1} + \cdots + c_y, tn_{aI} - (c_{x+1} + c_{x+2} + \cdots + c_{y+1}))$

- $A = a_1 + a_2 + \cdots + a_{x-1} + sn_{bI} + tn_{cI} - n_{aI}$
- $B = b_1 + b_2 + \cdots + b_{x-1} + n_{bI}$
- $C = c_1 + c_2 + \cdots + c_{x-1} + n_{cI}$

If this path is the string data of a stable component, then

(1) $\frac{A}{sB+tC} \geq \frac{a}{sb+tc}$
(2) *If* $\frac{A}{sB+tC} = \frac{a}{sb+tc}$ *and* $C > 0$*, then* $\frac{B}{C} \geq \frac{b}{c}$

Proof. Assume there is a Λ-module M with this string data, And assume the condition is violated. It suffices to show that M is not stable, which we will do by finding a submodule of M that violates stability.

Decompose M as in (3.5). By the definition of the socle filtration, the map

$$_1x_2 : B_x \oplus \cdots \oplus B_{y+1} \to A_x^s \oplus \cdots \oplus A_{y+1}^s$$

is injective. By the preprojective relation,

$$_2\bar{x}_1 \circ {_1x_2} : B_x \oplus \cdots \oplus B_{y+1} \to B_{x-1} \oplus \cdots \oplus B_y$$

is the zero map, so

$$\dim \operatorname{im} {_2\bar{x}_1} \leq s(a_x + \cdots + a_{y+1}) - (b_x + \cdots + b_{y+1}).$$

Since clearly $\dim \operatorname{im} {_2\bar{x}_1} \leq b_x + b_{x+1} + \cdots + b_y$, we see that $\dim \operatorname{im} {_2\bar{x}_1} \leq n_{bI}$. Similarly, $\dim \operatorname{im} {_3\bar{x}_1} \leq n_{cI}$.

Let $I_2 = \operatorname{im} {_2\bar{x}_1} \subset B_x \oplus \cdots \oplus B_y$ and $I_3 = \operatorname{im} {_3\bar{x}_1} \subset C_x \oplus \cdots \oplus C_y$ Consider the submodule S generated by

$$M_1 \oplus \cdots \oplus M_{x-1} \oplus I_2 \oplus I_3.$$

Then it is clear that $\dim S_2 = B, \dim S_3 = C$.

Consider

$$_1\bar{x}_2 \oplus {_1\bar{x}_3} : (\operatorname{im} {_2\bar{x}_1})^s \oplus (\operatorname{im} {_3\bar{x}_1})^t \to A_{x-1} \oplus \cdots \oplus A_y.$$

Since $\mathrm{im}\, {}_2x_1 \oplus {}_3x_1 \subset (\mathrm{im}\, {}_2\bar{x}_1)^s \oplus (\mathrm{im}\, {}_3\bar{x}_1)^s$ the preprojective relation implies that this has a kernel of dimension at least n_{aI}. So, the image of this map has dimension at most

$$sn_{bI} + tn_{cI} - n_{aI}.$$

Thus $\dim S_1 \leq a_1 + \cdots + a_{x-1} + sn_{bI} + tn_{cI} - n_{aI} = A$.

But then S violates stability, since either

$$\frac{\dim S_1}{s \dim S_2 + t \dim S_3} \leq \frac{A}{sB + tC} < \frac{a}{sb + tc},$$

or these are equal and

$$\frac{\dim S_2}{S_3} = \frac{B}{C} < \frac{b}{c}.$$

■

6. More examples

Example 6.1. Consider $s = t = 2$ and the path

$$1^3 2^3 1^3 2^2 3 1^4 2^5. \tag{6.2}$$

This passes all the conditions in Theorem 1.1. However, by the pre-projective relation and the fact that this is a socle filtration, the sub-quotient corresponding to the sub-path $1^3 2^2 3 1^4$ (shown in red) has the property that $\mathrm{im}({}_1x_2 + {}_1x_3)$ has dimension at most 2. This in turn implies the existence of a submodule

$$1^3 2^3 1^2 2^2 3,$$

and that violates stability. This is caught by Theorem 5.1 with $x = y = 2$.

Example 6.3. Again take $s = t = 2$, but now the path

$$\underline{1^4 2^3} 1^3 \underline{2^2 3} 1^4 \underline{2^2 3} 1^4 2^6 1^5 2^7.$$

By the preprojective relation, t he sub-quotient corresponding to $2^2 3 1^4 2^2 3 1^4$ has the property that $\dim \mathrm{im}(x_2 + x_3) \leq 4$. Thus the vectors corresponding to the underlined part of the path must general a sub-module violating stability. This is caught by Theorem 5.1 with $x = 2, y = 3$.

Example 6.4. Again take $s = t = 2$, but now the path

$$1^4 2^4 1^4 2^3 3 1^4 2^6 1^5 2^6.$$

Then the submodule corresponding to $2^3 3 1^4 2^6$ has the property that $\dim \mathrm{im}({}_2x_1) \leq 2$, which forces a submodule of the form

$$1^4 2^4 1^4 2^2 3 1^4$$

But now the sub-quotient $1^4 2^2 3 1^4$ has $\dim \mathrm{im}({}_1x_2 + {}_1x_3)) \leq 2$, forcing a submodule with data

$$1^4 2^4 1^2 2^2 3,$$

and now stability has been violated. The cases when $n_{bI} = sn_{aI} - (b_{x+1} + b_{x+2} + \cdots + b_{y+1})$ (or similarly with n_{cI}) catch this sort of two-step problem.

Example 6.5. For $s = 2, t = 1$, the smallest root where the number of paths satisfying both Theorems 1.1 and 5.1 is not the root multiplicity is $6\alpha_1 + 5\alpha_2 + 3\alpha_3$. Somewhat surprisingly, the correct multiplicity in 30 but there are 33 paths in this case. So, the first error is off by 3! The three paths that pass all the conditions but do not correspond to stable irreducible components are

$$11221123123123 \tag{6.6}$$

$$11212123123123 \tag{6.7}$$

$$11122211233123 \tag{6.8}$$

To see why these do not correspond to stable components, consider (6.6). Look at the red quotient module. The map ${}_1x_3$ has a 2 dimensional image by the definition of the socle filtration, so the preprojective relation ${}_3x_1 \circ {}_1x_3 = 0$ implies that the map ${}_3x_1$ is the zero map on this quotient. So, there is a quotient module Q isomorphic to $\mathbb{C}_3^3$, and a submodule P with data

112211**2121**2.

Now, the quotient module marked in bold implies the existence of a submodule P' with data

11221212

Denote the span of these four 1s by W. Then the map ${}_1x_3 : Q \to T_1/W$ has a one dimensional kernel K_1 by dimension count. Then $P' \oplus K$ violates stability.

The arguments for the other two cases are very similar: in both cases, there is a quotient Q isomorphic to $\mathbb{C}_3^3$ by essentially the same argument.

For (6.7), the corresponding submodule is

11**21212121**2,

and the bold sub-quotient implies the existence of a submodule with data

12121212.

Now the argument is the same.

For (6.8), the corresponding submodule is

11122211**21**2,

and the bold sub-quotient implies the existence of a submodule with data

11122212.

Now the argument is the same.

One may ask, why is the first example off by 3? It seems strange, but is just combinatorially hard to find a place where this sort of example happens. We are only looking at roots which are minimal in the sense that they can't be reflected to smaller roots. So, that means roots $a\alpha_1 + b\alpha_2 + c\alpha_3$ with $b \leq a, c \leq \frac{a}{2}$, and $a \leq b + \frac{c}{2}$. It is just not possible for combinatorial reasons to construct an example like that in Example 6.5 for a root satisfying these conditions and smaller than $6\alpha_1 + 5\alpha_2 + 3\alpha_3$, and then suddenly there is quite a bit of freedom. However, it is worth considered roots which are not minimal. Our method works just fine for such roots, even if they can be reflected to a simpler case. We are then able to find a more minimal looking example, where the error is only by 1.

Example 6.9. Take $s = 2, t = 1$ and the root $4\alpha_1 + 3\alpha_2 + 3\alpha_3$. This can be reflected to $4\alpha_1 + 3\alpha_2 + \alpha_3$ and then to $3\alpha_1 + 3\alpha_2 + \alpha_3$, a root which we know has multiplicity 3. However, let's do the calculation with our methods directly on $4\alpha_1 + 3\alpha_2 + 3\alpha_3$. Then a total of 5 paths pass the conditions in Theorem 1.1:

1 1 2 3 3 1 2 1 2 3
1 1 1 2 2 3 3 1 2 3
1 1 1 1 2 2 2 3 3 3
1 1 2 3 1 2 3 1 2 3
1 1 2 3 3 1 1 2 2 3

The path 1123311223 is ruled out by Theorem 5.1 with $x = y = 1$. Here $n_{aI=2}$, which is just a_2; $n_{bI} = 1$, since $\min\{b_1, a_2 - 2b_2\} = b_1 = 1$; $n_{cI} = 1$ because $\min\{c_1, a_2 - c_2\} = a_2 - c_2 = 1$. The condition says

$$\frac{2n_{bI} + n_{cI} - n_{a_I}}{2n_{bI} + n_{cI}} \geq \frac{4}{2 \times 3 + 3},$$

which gives $\frac{1}{3} \geq \frac{4}{9}$, which is false.

The other path which does not correspond to a stable component is 1123123123. One can see that this module is not stable by the same argument as in Example 6.5, and in fact the situation is a little simpler here.

7. Computational data

In the cases $s = t = 2$ and $s = 2, t = 1$ we computed our estimates in many examples using Python. The code can be found at [Chan]. The actual multiplicities for $s = 2, t = 1$ can be found in [Kac90]. The multiplicities in the case $s = t = 2$ were calculated by Alex Feingold by modifying mathematica code originally written by Stephen Miller implementing a version of the Peterson algorithm to calculate the multiplicities, see [Pet83]. The results are given in Figures 7.2 and 7.1. These seem to validate our belief that these upper bounds are pretty good.

Note that, for the case $s = t = 2$, the symmetry of the Dynkin diagram implies that for any a, k, ℓ, the roots $a\alpha_1 + k\alpha_2 + \ell\alpha_3$ and $a\alpha_1 + \ell\alpha_2 + k\alpha_3$ have the same multiplicity. However, our method breaks this symmetry, and does not always give the same estimates in these cases. See for example the data for the roots $9\alpha_1 + 8\alpha_2 + 7\alpha_3$ and $9\alpha_1 + 7\alpha_2 + 8\alpha_3$. The symmetry breaking only coms into effect what a and $k+\ell$ are not relatively prime, so that Dyck paths that touch the diagonal are possible. So, for instance, our estimates are the same for $9\alpha_1 + 9\alpha_2 + 7\alpha_3$ and $9\alpha_1 + 7\alpha_2 + 9\alpha_3$.

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Case: (A,B,C)	Actual multiplicity	Bound using Theorems 1.1 and 5.1	Bound using just Theorem 1.1
(1, 1, 1)	1	1	1
(2, 2, 1)	3	3	3
(2, 1, 2)	3	3	3
(3, 3, 1)	5	5	5
(4, 4, 1)	10	10	10
(3, 2, 2)	10	10	10
(3, 3, 2)	12	12	12
(3, 2, 3)	12	12	12
(5, 5, 1)	16	16	18
(6, 6, 1)	28	28	34
(4, 3, 2)	25	25	25
(7, 7, 1)	43	43	61
(4, 3, 3)	46	46	46
(8, 8, 1)	70	70	114
(5, 4, 2)	61	61	64
(4, 4, 3)	58	58	58
(4, 3, 4)	58	58	58
(5, 5, 2)	73	73	76
(9, 9, 1)	105	106	204
(10, 10, 1)	161	165	377
(6, 5, 2)	133	133	147
(5, 4, 3)	153	153	156
(5, 3, 4)	153	153	156
(11, 11, 1)	236	244	680
(5, 5, 3)	181	182	185
(5, 3, 5)	181	182	185
(7, 6, 2)	283	283	340
(5, 4, 4)	262	262	265
(7, 7, 2)	335	336	392
(5, 5, 4)	307	308	311
(5, 4, 5)	307	308	311
(6, 5, 3)	439	439	458
(8, 7, 2)	565	567	744
(9, 8, 2)	1100	1108	1612
(6, 5, 4)	969	970	990
(10, 7, 6)	251656	251911	284878
(9, 8, 7)	273917	275221	281488
(9, 7, 8)	273917	275046	281363
(9, 9, 7)	303947	306371	311847
(9, 7, 9)	303947	306371	311847

FIGURE 7.1. Root multiplicities and our estimates for $s = t = 2$. The table begins by looking at small roots, then jumps to some of the largest roots we were able to work with.

Case: (A,B,C)	Actual multiplicity	Bound using Theorems 1.1 and 5.1	Bound using just Theorem 1.1
(2, 2, 1)	2	2	2
(3, 3, 1)	3	3	3
(4, 3, 2)	5	5	6
(4, 4, 1)	5	5	6
(5, 5, 1)	7	7	9
(5, 4, 2)	11	11	15
(6, 6, 1)	11	11	16
(5, 5, 2)	15	15	18
(7, 7, 1)	15	15	24
(6, 5, 2)	22	22	34
(8, 8, 1)	22	22	39
(6, 5, 3)	30	33	46
(9, 9, 1)	30	30	61
(7, 6, 2)	42	42	72
(10, 10, 1)	42	42	96
(7, 7, 2)	56	56	79
(11, 11, 1)	56	56	148
(7, 6, 3)	77	83	121
(8, 7, 2)	77	77	146
(12, 12, 1)	77	77	233
(7, 7, 3)	101	101	134
(9, 8, 2)	135	137	283
(8, 7, 3)	176	187	296
(9, 9, 2)	176	176	287
(8, 7, 4)	231	253	379
(8, 8, 3)	231	233	316
(10, 9, 2)	231	235	531
(9, 7, 4)	297	317	725
(9, 8, 3)	385	410	682
(11, 10, 2)	385	399	974
(9, 8, 3)	385	410	682
(11, 11, 2)	490	499	934
(9, 8, 4)	627	674	1062
(9, 9, 4)	792	807	1107
(10, 9, 3)	792	839	1498
(10, 8, 5)	1002	1218	2335
(10, 9, 4)	1574	1656	2754
(11, 10, 3)	1574	1673	3161
(10, 9, 5)	1957	2167	3404
(11, 9, 4)	1957	2029	5113
(11, 11, 3)	1956	2000	3134
(11, 9, 5)	3007	3492	6942
(11, 10, 4)	3713	3912	6776

FIGURE 7.2. Root Multiplicity Data: s=2, t=1.

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