Loyola University Chicago
Math 475, Fall 2014

HW 1

Problem 1. For $x = (x_1, x_2) \in \mathbb{R}^2$ and $p \in (0, \infty)$, define

$$\|x\|_p = (|x_1|^p + |x_2|^p)^{1/p}.$$

So, for example, $\|x\|_{1/2} = \left(\sqrt{|x_1|} + \sqrt{|x_2|}\right)^2$. Also, for $x = (x_1, x_2) \in \mathbb{R}^2$, define

$$\|x\|_\infty = \max\{|x_1|, |x_2|\}.$$

(a) Check if $\|\cdot\|_{1/2}$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ are norms on $\mathbb{R}^2$. If yes, prove it. If no, give an example showing why not.

(b) For $p = 1/2, 2, \infty$, sketch the unit sphere $\{x \in \mathbb{R}^2 \mid \|x\|_p = 1\}$.

(c) Show that, for every $x \in \mathbb{R}^2$, $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

(d) Find $m, M \in \mathbb{R}$ so that, for every $x \in \mathbb{R}^2$,

$$m\|x\|_2 \le \|x\|_\infty \le M\|x\|_2.$$

Repeat with $\|\cdot\|_1$ and $\|\cdot\|_\infty$.

(e) Let $\mathbb{B}_\infty = \{x \in \mathbb{R}^2 \mid \|x\|_\infty \le 1\}$ be the unit ball in $\mathbb{R}^2$ centered at 0 for the norm $\|\cdot\|_\infty$. Let P be "the projection" from $\mathbb{R}^2$ onto $\mathbb{B}_\infty$, i.e., for each $x \in \mathbb{R}^2$ let $P(x)$ be the point y that minimizes or the set of points y that minimize $\|x - y\|_\infty$ over all $y \in \mathbb{B}_\infty$. Find $P((1.5, 0))$, $P((2, 2))$, $P((2, 1.5))$, $P((3, 0))$

Problem 2. You know the following facts:

A: For every sequence of points in $[a, b]$ there exists a convergent subsequence.

B: $f : [a, b] \to \mathbb{R}$ is continuous if and only if for every convergent sequence of points $x_i \in [a, b]$, one has

$$\lim_{i\to\infty} f(x_i) = f\left(\lim_{i\to\infty} x_i\right).$$

C: Given a bounded set $S \subset \mathbb{R}$, if $s = \sup S$ then for every $\varepsilon > 0$ there exists $x \in S$ such that $x > s - \varepsilon$.

Let $f : [a, b] \to \mathbb{R}$ be continuous. Using the facts above, prove the following

(a) f is bounded above: there exists $K \in \mathbb{R}$ so that, for every $x \in [a, b]$, $f(x) < K$.

(b) f attains its maximum on $[a, b]$: there exists $x \in [a, b]$ so that $f(x) = \sup\{f(x) \mid x \in [a, b]\}$.

(c) f is uniformly continuous: for every $\varepsilon > 0$ there exists $\delta > 0$ so that, for every $x, y \in [a, b]$ with $|x - y| < \delta$ one has $|f(x) - f(y)| < \varepsilon$.

More problems will come soon...