Loyola University Chicago **HW 8**
Math 475, Fall 2014

Book Problems. (MacCluer) Read, and re-read Appendix A.

Problem 0. Let X be a Banach space with norm $\|\cdot\|_X$ and let Y be a Banach space with norm $\|\cdot\|_Y$. Prove that

$$\|(x,y)\| = \max\{\|x\|_X, \|y\|_Y\}$$

defines a norm on $X \times Y$ and that with this norm, $X \times Y$ is a Banach space.

Problem 1. Let $S \subset \mathbb{R}^n$ and consider the support function $\sigma_S : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\sigma_S(x) = \sup_{s \in S} s \cdot x.$$

Prove, without invoking the Principle of Uniform Boundedness, that if σ_S is finite-valued then S is bounded. *(It may help to remember that in $\mathbb{R}^n$, the unit ball is compact...)*

Problem 2. Let X be a Banach space with norm $\|\cdot\|$. Let $\phi : X \to \mathbb{R}$ be a nonzero linear functional. Prove that

$$\|x\|_\phi := \|x\| + |\phi(x)|$$

defines a different norm on X. Prove that if ϕ is a bounded linear functional, then $\|\cdot\|_\phi$ is equivalent to $\|\cdot\|$.

Problem 3. For an arbitrary $\phi \in c^*$, let $x_i = \phi(e_i)$, where e_i is the usual i-th unit vector; let $x = (x_1, x_2, x_3, \ldots)$; and let $r = \phi(\mathbf{1})$, where $\mathbf{1} = (1, 1, 1, \ldots)$. Let $T : \mathbb{R} \times l^1 \to c^*$ be defined by, for every $y \in c$,

$$T(r,x)(y) = r \lim_{i\to\infty} y_i + \sum_{i=1}^{\infty} x_i y_i.$$

Show that $T(r,x) = \phi$. *(Slight chance I have done this in class and I do not remember; if so, you can skip it...)*

Problem 4. Suppose that X is a normed vector space and $T : X \to X$ is a function that has closed graph. Is it true that T maps closed sets to closed sets? Is it true if T is also linear?