Loyola University Chicago
Math 161, Spring 2010

Final Exam

Name (print): ____________________ **Signature:** ____________________

Please do not start working until instructed to do so.

You have 2 hours.

No calculators, iPhone's, laptops, or any other devices that do more than show time.

You must show your work to receive full credit.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Problem 8. ________

Total. ________

Problem 1. *(32 points total)* Find the following limits, derivatives, and integrals:

a. *(4 points)* $\lim_{x\to-\infty} \dfrac{x^3 - 5x^7 + 8x}{x^4 + 9 + 15x^7}$

Solution: divide numerator and denominator by x^7:

$$\lim_{x\to-\infty} \frac{x^3 - 5x^7 + 8x}{x^4 + 9 + 15x^7} = \lim_{x\to-\infty} \frac{\frac{1}{x^4} - 5 + 8\frac{1}{x^6}}{\frac{1}{x^2} + \frac{9}{x^7} + 15} = \frac{-5}{15} = -\frac{1}{3}$$

b. *(4 points)* $\dfrac{d}{dx}\left(\cos(5x^2) + 3^x \ln x\right)^6.$

Solution:

$$\frac{d}{dx}\left(\cos(5x^2) + 3^x \ln x\right)^6 = 6\left(\cos(5x^2) + 3^x \ln x\right)^5 \left(-\sin(5x^2)10x + \ln 3\, 3^x \ln x + 3^x \frac{1}{x}\right)$$

c. *(4 points)* $\lim_{x\to\infty} \dfrac{1 + \sqrt{x-3}}{x}.$

Solution:

$$\lim_{x\to\infty} \frac{1 + \sqrt{x-3}}{x} = \lim_{x\to\infty} \frac{1}{x} + \frac{\sqrt{x-3}}{x} = \lim_{x\to\infty} \frac{1}{x} + \sqrt{\frac{1}{x} - \frac{3}{x^2}} = 0$$

Alternatively, use L'Hopitals rule:

$$\lim_{x\to\infty} \frac{1 + \sqrt{x-3}}{x} = \lim_{x\to\infty} \frac{0 + \frac{1}{2\sqrt{x-3}}}{1} = \lim_{x\to\infty} \frac{1}{2\sqrt{x-3}} = 0$$

d. *(4 points)* $\lim_{t\to 0^-} 1 + \dfrac{2}{\sqrt[3]{t}}$

Solution:

$$\lim_{t\to 0^-} 1 + \frac{2}{\sqrt[3]{t}} = 1 + \lim_{t\to 0^-} \frac{2}{\sqrt[3]{t}} = -\infty$$

because when $t \to 0^-$, $\sqrt[3]{t} \to 0$ while $\sqrt[3]{t} < 0$.

e. *(4 points)* $\dfrac{d}{dx}\displaystyle\int_{e^x}^{\tan^{-1}(x)} (x+1)^{777}\, dt$

Solution: Fundamental Theorem of Calculus and chain rule

$$\frac{d}{dx}\int_{e^x}^{\tan^{-1}(x)} (x+1)^{777}\, dt = \left(\tan^{-1}(x)+1\right)^{777} - (e^x+1)^{777}\, e^x$$

f. *(4 points)* $\displaystyle\int \left(\frac{5}{\sqrt{1-x^2}} - x^7 + 11\right) dx$

Solution:

$$\int \left(\frac{5}{\sqrt{1-x^2}} - x^7 + 11\right) dx = 5\sin^{-1}(x) - \frac{1}{8}x^8 + 11x + C$$

g. *(4 points)* $\displaystyle\int_0^6 \left(x + \sqrt{36-x^2}\right) dx.$

Solution:

$$\int_0^6 \left(x + \sqrt{36-x^2}\right) dx = \int_0^6 x\, dx + \int_0^6 \sqrt{36-x^2}\, dx = \frac{1}{2}6\cdot 6 + \frac{1}{4}\pi 6^2 = 18 + 9\pi$$

h. *(4 points)* $\displaystyle\int \frac{x}{\sqrt{x+7}}\, dx$

Solution: set $u = x+7$, so that $du = dx$ and $x = u-7$. Get

$$\int \frac{x}{\sqrt{x+7}}\, dx = \int \frac{u-7}{\sqrt{u}}\, du = \int u^{1/2} - 7u^{-1/2}\, du = \frac{2}{3}u^{3/2} - 14u^{1/2} + C = \frac{2}{3}(x+7)^{3/2} - 14(x+7)^{1/2} + C$$

Problem 2. *(8 points total)* Use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{x^2+1}$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0} \frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} \\ &= \lim_{h\to 0} \frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)} = \lim_{h\to 0} \frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}} \\ &= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}} \end{aligned}$$

Problem 3. *(8 points total)* Find all horizontal and all vertical asymptotes of the function $f(x) = \dfrac{2-e^x}{2+e^x}$

Solution: there is no vertical asymptotes because the function is continuous, and that is because $2+e^x$ is always positive, in fact always greater than 2. For horizontal asymptotes, do this:

$$\lim_{x\to-\infty} \frac{2-e^x}{2+e^x} = \frac{2-\lim_{x\to-\infty} e^x}{2+\lim_{x\to-\infty} e^x} = \frac{2-0}{2+0} = 1$$

$$\lim_{x\to\infty} \frac{2-e^x}{2+e^x} = \lim_{x\to\infty} \frac{\frac{2}{e^x}-1}{\frac{2}{e^x}+1} = \frac{-1}{1} = -1$$

There are two horizontal asymptotes: $y=1$ and $y=-1$.

Problem 4. *(14 points total)* Let $g(x) = \dfrac{1}{4}x^4 - x^2$.

a. *(6 pts)* Find the critical points for this function. For each critical point, determine whether it is a local minimum, local maximum, or neither.

Solution: the function is differentiable so the only critical points are where $f'(x) = 0$.

$$f'(x) = x^3 - 2x = x(x^2 - 2) = x(x - \sqrt{2})(x + \sqrt{2})$$

so the critical points are $x = 0$, $x = \sqrt{2}$, $x = -\sqrt{2}$. One way to check if they are local minima or maxima is to look at the second derivative.

$$f''(x) = 3x^2 - 2$$

Then $f(0) = -2 < 0$, $f(\sqrt{2}) = f(-\sqrt{2}) = 4 > 0$. So $x = 0$ is a local maximum, $x = \sqrt{2}$, $x = -\sqrt{2}$ are local minima.

b. *(4 pts)* Identify the intervals on which g is concave up and concave down.

Solution:

$$f''(x) = 3(x^2 - 2/3) = 3(x - \sqrt{2/3})(x + \sqrt{2/3})$$

Then $f''(x) < 0$ in $(-\sqrt{2/3}, \sqrt{2/3})$; $f''(x) > 0$ on $(-\infty, -\sqrt{2/3})$ and $(\sqrt{2/3}, \infty)$. Consequently, f is concave down on $(-\sqrt{2/3}, \sqrt{2/3})$; f is concave up on $(-\infty, -\sqrt{2/3})$ and $(\sqrt{2/3}, \infty)$.

c. *(4 pts)* Find the absolute minimum and the absolute maximum of $g(x)$ on the interval $[-1, 3]$.

Solution: there are two critical points in $[0, 3]$: $x = 0$ and $x = \sqrt{2}$. Compare values at these points and at the endpoints:

$$f(-1) = -\frac{3}{4}, \quad f(0) = 0, \quad f(\sqrt{2}) = -1, \quad f(3) = 11\frac{1}{4}$$

The absolute minimum is at $x = \sqrt{2}$, the absolute maximum is at $x = 3$.

Problem 5. *(8 points)* Consider the function

$$f(x) = \ln x - x.$$

a. *(4 pts)* Write the left-endpoint Riemann sum for this function on the interval $[1, 3]$ with $n = 4$ subintervals. *(You do not need to evaluate the sum.)*

Solution: $\Delta x = \frac{3-1}{4} = .5$, *then* $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, $x_4 = 3$. *Left-endpoint Riemann sum is*

$$\Delta x(f(x_0) + f(x_1) + f(x_2) + f(x_3)) = 0.5\,(\ln 1 - 1\ln 1.5 - 1.5 + \ln 2 - 2 + \ln 2.5 - 2.5)$$

b. *(4 pts)* Circle the correct statement and provide a brief explanation of your answer.

- The left-endpoint Riemann sum is an underestimate of $\int_1^3 \ln x - x\,dx$.
- The left-endpoint Riemann sum is an overestimate of $\int_1^3 \ln x - x\,dx$.
- Neither of the two statements above is true.

Solution: check if the funciton is decreasing or increasing or neither on $[1, 3]$.

$$f'(x) = \frac{1}{x} - 1$$

and $\frac{1}{x} - 1 < 0$ *because* $\frac{1}{x} < 1$, $1 < x$ *when* x *is in* $(1, 3)$. *So* f *is decreasing, hence the left-endpoint sum is an overestimate.*

Problem 6. *(10 points)* Laws of physics determine that the acceleration of a rocket cart is given by the formula

$$a(t) = 2t - 5$$

for $t \geq 0$. The velocity of the rocket cart was measured at $t = 0$ to be $v(t) = 6$ meters/second.

a. *(5 pts)* Find the formula for the velocity of the rocket cart at time $t \geq 0$.

Solution: $a(t) = v'(t)$ *so*

$$v(t) = \int a(t)\,dt = \int 2t - 5\,dt = t^2 - 5t + C$$

Determine C *using* $v(0) = 6$*:* $v(0) = 0^2 - 5 \cdot 0 + C = 6$*, so* $C = 6$*. Thus*

$$v(t) = t^2 - 5t + 6$$

b. *(5 pts)* Find the total distance traveled by the cart between $t = 0$ and $t = 3$.

Solution: $v(t) = t^2 - 5t + 6 = (t-2)(t-3)$ *so* $v(t) \geq 0$ *on* $[0, 2]$ *and* $v(t) < 0$ *on* $[2, 3]$*. Then the total distance is*

$$\int_0^3 |v(t)|\,dt = \int_0^2 v(t)\,dt + \int_2^3 -v(t)\,dt = \int_0^2 t^2 - 5t + 6\,dt - \int_2^3 t^2 - 5t + 6\,dt = \frac{29}{6}$$

Problem 7.*(10 points)* A small helium balloon is rising at the rate of 8 ft/sec, a horizontal distance of 12 feet from a 20 ft. lamppost. At what rate is the shadow of the balloon moving along the ground when the balloon is 5 feet above the ground?

Solution: let h be the height of the balloon. Let x be the distance from the lamppost to the shadow of the balloon. We know $dh/dt = 5$, we need to find dx/dt. Use similar triangles to relate x to h:

$$\frac{20}{x} = \frac{h}{x-12} \quad \textit{so} \quad x = \frac{240}{20-h}$$

Differentiate with respect to t, get

$$\frac{dx}{dt} = -\frac{240}{(20-h)^2}\left(-\frac{dh}{dt}\right)$$

Plug in $h = 5$, $dh/dt = 5$, get

$$\frac{dx}{dt} = \frac{128}{15}$$

Problem 8. *(10 points total)* There are two points on the ellipse

$$5x^2 - 6xy + 5y^2 = 4$$

with the x-coordinate equal to 1. Find where the tangent lines to the ellipse, at these two points, intersect.

Solution: set $x = 1$, *get* $5 - 6y + 5y^2 = 4$, *so* $5y^2 - 6y + 1 = (5y - 1)(y - 1) = 0$, *so* $y = 1$ *or* $y = 1/5$. *The two points on the ellipse are* $(1, 1)$ *and* $(1, 1/5)$. *Differentiate implicitly, get*

$$10x - 6y - 6xy' + 10yy' = 0$$

At the point $(1, 1)$ *get*

$$10 - 6 - 6y' + 10y' = 0, \quad 4 = -4y', \quad y' = -1$$

and the tangent line at $(1, 1)$ *is* $y - 1 = -1(x - 1)$, *so* $y = -x + 2$. *At the point* $(1, 1/5)$ *get*

$$10 - 6/5 - 6y' + 10/5y' = 0, \quad 8\frac{4}{5} = 4y', \quad y' = 2\frac{1}{5}$$

and the tangent line at $(1, 1/5)$ *is* $y - 1/5 = 2\frac{1}{5}(x - 1)$, *so* $y = 2\frac{1}{5}x - 2$. *Intersect the two tangent lines to get*

$$-x + 2 = 2\frac{1}{5}x - 2, \quad 4 = 3\frac{1}{5}x, \quad x = 5/4$$

Then $y = -x + 2 = -5/4 + 2 = 3/4$. *The point where the tangent lines intersect is then*

$$(5/4, 3/4)$$