

Name (print): _____ Signature: _____

Please do not start working until instructed to do so.

You have 2 hours.

You must show your work to receive full credit.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. _____

Problem 2. _____

Problem 3. _____

Problem 4. _____

Problem 5. _____

Problem 6. _____

Problem 7. _____

Problem 8. _____

Problem 9. _____

Total. _____

Problem 1. (16 points total) Find the following limits and derivatives

a. (4 points) $\lim_{x \rightarrow \infty} \frac{7x - 11x^2 + x^3}{4x^3 + 5x^2}$

Answer/solution: $\frac{1}{4}$

b. (4 points) $\frac{d}{dx} \tan\left(\frac{x^3}{5^x}\right)$.

Answer/solution:

$$\frac{1}{\cos\left(\frac{x^3}{5^x}\right)} \frac{3x^2 5^x - x^3 \ln 5 \cdot 5^x}{(5^x)^2}$$

c. (4 points) $\lim_{x \rightarrow 0} \frac{e^{cx} - 1}{x}$ where $c \neq 0$ is a constant.

Answer/solution:

$$\lim_{x \rightarrow 0} \frac{e^{cx} - 1}{x} = \lim_{x \rightarrow 0} \frac{ce^{cx}}{1} = c$$

d. (4 points) $\lim_{x \rightarrow 2^+} \frac{x^2 - 3x + 2}{x^2 - x - 2}$

Answer/solution:

$$\lim_{x \rightarrow 2^+} \frac{x^2 - 3x + 2}{x^2 - x - 2} = \lim_{x \rightarrow 2^+} \frac{(x-1)(x-2)}{(x+1)(x-2)} = \frac{1}{3}$$

Problem 2. (12 points) Find the following integrals

a. (4 points) $\int \left(1 - 3x^7 + \frac{3}{\sqrt{1-x^2}} - 3^x \right) dx$

Answer/solution:

$$x - \frac{3}{8}x^8 + 3 \sin^{-1} x - \frac{1}{\ln 3} 3^x + C$$

b. (4 points) $\int_{-3}^0 \left(\sqrt{9-x^2} - x \right) dx.$

Answer/solution:

$$\frac{9}{4}\Pi + \frac{9}{2}$$

c. (4 points) $\int \frac{2e^{3x}}{4 + e^{3x}} dx$

Answer/solution:

$$\frac{2}{3} \ln(4 + e^{3x}) + C$$

Problem 3. (8 points total) Use the definition of the derivative to find $f'(x)$ when $f(x) = \sqrt{1+x^2}$.

Answer/solution:

Problem 4. (8 points total) Let $f(x) = \int_1^x \frac{t^2 + 12t + 27}{2 + e^t} dt$. At what point x does f have a local minimum?

Answer/solution:

$$f'(x) = \frac{x^2 + 12x + 27}{2 + e^x} = \frac{(x+3)(x+9)}{2 + e^x}$$

The derivative is 0 at $x = -3$ and at that point it changes from negative to positive. So, this is a local minimum.

Problem 5. (16 points total) An object moves following the law of motion

$$s(t) = \frac{t^2}{8} + \ln(1+t)$$

where $t \geq 0$ is measured in seconds and s is in meters.

a. (4 points) Find the velocity at time t .

Answer/solution:

$$v(t) = \frac{t}{4} + \frac{1}{1+t}$$

b. (4 points) When is the velocity positive?

Answer/solution: all $t \geq 0$

c. (4 points) Find the acceleration at time t .

Answer/solution:

$$a(t) = \frac{1}{4} - \frac{1}{(1+t)^2}$$

d. (4 points) Find the time $t \geq 0$ at which the velocity is minimal.

Answer/solution: $t = 1$

Problem 6. (8 points total) Find the highest point, in the first quadrant (so with $x \geq 0$ and $y \geq 0$), on the curve

$$x^3 + y^3 - 15xy = 0.$$

Answer/solution: find point on curve where $y' = 0$. Implicit differentiation:

$$3x^2 + 3y^2y' - 15y - 15xy' = 0, \quad y' = \frac{15y - 3x^2}{3y^2 - 15x}, \quad y' = 0 \text{ gives } y = \frac{1}{5}x^2$$

Plug $y = \frac{1}{5}x^2$ into equation of curve, get

$$x^3 + \frac{1}{125}x^6 - 3x^3 = 0, \quad \frac{1}{125}x^3 = 2, \quad x = 5\sqrt[3]{2}$$

plug for y , get $y = 5 \cdot 2^{3/2}$.

Problem 7. (8 points total) Find $\lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{7x}$. Show all the details of your work!

Answer/solution:

$$e^{14}$$

Here are the details:

$$\begin{aligned} \lim_{x \rightarrow \infty} \left(1 + \frac{2}{x}\right)^{7x} &= \lim_{x \rightarrow \infty} e^{\ln\left(1 + \frac{2}{x}\right)^{7x}} = \lim_{x \rightarrow \infty} e^{7x \ln\left(1 + \frac{2}{x}\right)} = e^{\lim_{x \rightarrow \infty} 7x \ln\left(1 + \frac{2}{x}\right)} \\ &= e^{\lim_{x \rightarrow \infty} \frac{\ln\left(1 + \frac{2}{x}\right)}{\frac{1}{7x}}} = e^{\lim_{x \rightarrow \infty} \frac{\frac{1}{\left(1 + \frac{2}{x}\right)} \left(-\frac{2}{x^2}\right)}{-\frac{1}{7x^2}}} = e^{\lim_{x \rightarrow \infty} \frac{14}{\left(1 + \frac{2}{x}\right)}} \\ &= e^{14} \end{aligned}$$

Problem 8. (12 points) A 216 m² rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

Answer/solution: one side is 18, the other is 12. Total length consists of 2 sides of length 18 and 3 sides of length 12. Total is 72.

Details: the fence will be made from two segments of length x and three segments of length y . Total fence is $2x + 3y$. Total area is $xy = 216$ so $y = 216/x$. Then the total fence is

$$f(x) = 2x + \frac{648}{x}$$

and this is the quantity that needs to be minimized. Then

$$f'(x) = 2 - \frac{648}{x^2}, \quad f'(x) = 0 \implies 2 = \frac{648}{x^2}, \quad x^2 = 324, \quad x = 18.$$

Then $y = 216/18 = 12$.

Problem 9. (12 points) A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

a. (6 points) How fast is the top of the ladder sliding down the wall then?

Answer/solution: $h = \sqrt{13^2 - x^2}$,

$$\frac{dh}{dt} = \frac{-2x}{2\sqrt{169 - x^2}} \frac{dx}{dt} = \frac{-12}{\sqrt{169 - 144}} 5 = -12$$

b. (6 points) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

Answer/solution: $A = .5xh$,

$$\frac{dA}{dt} = \frac{1}{2} \left(\frac{dx}{dt} h + x \frac{dh}{dt} \right) = \frac{1}{2} (5 \cdot 5 + 12 \cdot (-12))$$