Loyola University Chicago
Math 161, Fall 2010

Final Exam

Name (print): ______________________ **Signature:** ______________

Please do not start working until instructed to do so.

You have 2 hours.

No calculators, iPhone's, laptops, or any other devices that do more than show time.

You must show your work to receive full credit.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Problem 8. ________

Problem 9. ________

Total. ________

Problem 1. *(32 points total)* Find the following limits, derivatives, and integrals. Show your work. Put a box around $\boxed{\text{your final answer}}$.

a. *(4 points)* $\displaystyle\lim_{x\to\infty} \frac{11x^2 + 4x^7}{x^4 + 9 + 15x^7}$

Solution:

$$\lim_{x\to\infty} \frac{11x^2 + 4x^7}{x^4 + 9 + 15x^7} = \lim_{x\to\infty} \frac{11x^2 + 4x^7}{x^4 + 9 + 15x^7} \frac{x^{-7}}{x^{-7}} = \lim_{x\to\infty} \frac{11x^{-5} + 4}{x^{-3} + 9x^{-7} + 15} = \frac{4}{15}$$

b. *(4 points)* $\displaystyle\lim_{x\to 2} \frac{x^2 + x - 6}{2 - 3x + x^2}.$

Solution:

$$\lim_{x\to 2} \frac{x^2 + x - 6}{2 - 3x + x^2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = 5$$

c. *(4 points)* $\displaystyle\lim_{y\to 0} \frac{1 - \cos(4y)}{y^2}.$

Solution:

$$\lim_{y\to 0} \frac{1 - \cos(4y)}{y^2} = \lim_{y\to 0} \frac{4\sin(4y)}{2y} = \lim_{y\to 0} \frac{16\cos(4y)}{2} = 8$$

d. *(4 points)* $\displaystyle\frac{d}{dx}\left(e^{3x}\arcsin(x)\right)$

Solution:

$$\frac{d}{dx}\left(e^{3x}\arcsin(x)\right) = 3e^{3x}\arcsin(x) + e^{3x}\frac{1}{\sqrt{1-x^2}}$$

e. *(4 points)* $\dfrac{d}{dx}\ln\left(\cos(\ln x)\right)$

Solution:

$$\frac{d}{dx}\ln\left(\cos(\ln x)\right) = \frac{1}{\cos(\ln x)}\frac{d}{dx}\left(\cos(\ln x)\right) = \frac{1}{\cos(\ln x)}\left(-\sin(\ln x)\right)\frac{d}{dx}\ln x = -\frac{1}{\cos(\ln x)}\sin(\ln x)\frac{1}{x}$$

f. *(4 points)* $\displaystyle\int 5 + x^3 - \frac{2}{1+x^2}\,dx$

Solution:

$$\int 5 + x^3 - \frac{2}{1+x^2}\,dx = 5x + \frac{1}{4}x^4 - 2\arctan(x) + C$$

g. *(4 points)* $\displaystyle\int_{-4}^{4}\left(\sqrt{16-z^2} - 7\right)\,dz.$

Solution:

$$\int_{-4}^{4}\left(\sqrt{16-z^2} - 7\right)\,dz = \int_{-4}^{4}\sqrt{16-z^2}\,dz - \int_{-4}^{4} 7\,dz = \frac{1}{2}\pi 4^2 - 7\cdot 8 = 8\pi - 56$$

h. *(4 points)* $\displaystyle\int \frac{2x+3}{2x^2+6x-1}\,dx$

Solution: $u = 2x^2 + 6x - 1,\ du = (4x+4)dx = 2(2x+3)dx,$

$$\int \frac{2x+3}{2x^2+6x-1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x^2+6x-1| + C$$

Problem 2. *(8 points total)* Use <u>the definition of the derivative</u> to find $f'(x)$ when $f(x) = \dfrac{x}{x+2}$.

Solution:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h} = \lim_{h\to 0} \frac{(x+h)(x+2) - x(x+h+2)}{h(x+2)(x+h+2)} \\
&= \lim_{h\to 0} \frac{x^2+2x+hx+2h-x^2-xh-2x}{h(x+2)(x+h+2)} = \lim_{h\to 0} \frac{2h}{h(x+2)(x+h+2)} = \\
&= \lim_{h\to 0} \frac{2}{(x+2)(x+h+2)} = \frac{2}{(x+2)^2}
\end{aligned}
$$

Problem 3. *(8 points total)* Find the interval (or intervals) where $F(x)$ is increasing and the interval (or intervals) where $F(x)$ is decreasing.

$$F(x) = \int_1^{x^2-3x} \frac{e^{3t}}{1+t^2}\,dt$$

Solution: F is increasing where $F' > 0$, F is decreasing where $F' < 0$.

$$F'(x) = \frac{d}{dx}\int_1^{x^2-3x} \frac{e^{3t}}{1+t^2}\,dt = \frac{e^{3(x^2-3x)}}{1+(x^2-3x)^2}\frac{d}{dx}(x^2-3x) = \frac{e^{3(x^2-3x)}}{1+(x^2-3x)^2}(2x-3)$$

The big fraction in $F'(x)$ is always positive, so the sign of $F'(x)$ is the same as the sign of $2x-3$. Hence $F'(x) > 0$ if $x > 3/2$, $F'(x) < 0$ if $x < 3/2$.

Answer: F increasing on $(3/2, \infty)$, F decreasing on $(-\infty, 3/2)$.

Problem 4. *(8 points total)* Let $g(x) = xe^{x^2}$. Find the interval (or intervals) where $g(x)$ is concave up and the interval (or intervals) where $g(x)$ is concave down.

Solution: g concave up where $g'' > 0$, g concave down where $g'' < 0$.

$$g'(x) = e^{x^2} + xe^{x^2}2x = e^{x^2}(1 + 2x^2)$$

$$g''(x) = e^{x^2}2x(1 + 2x^2) + e^{x^2}(4x) = e^{x^2}(2x + 4x^3 + 4x) = e^{x^2}x(4x^2 + 6)$$

Since $e^{x^2} > 0$, $4x^2 + 6 > 0$ always, sign of g'' depends on x. So $g''(x) > 0$ if $x > 0$, $g''(x) < 0$ if $x < 0$.

Answer: the function is concave up if $x > 0$, concave down if $x < 0$.

Problem 5. *(8 points)* Find the equation of the tangent line, at the point $(-2, -1)$, to the curve

$$3\left(x^2 + y^2\right)^2 = 25\left(x^2 - y^2\right).$$

Solution: Implicit differentiation:

$$6(x^2 + y^2)(2x + 2yy') = 25(2x - 2yy')$$

Set $x = -2$, $y = -1$, solve for y':

$$6(4+1)(-4-2y') = 25(-4+2y'),\quad 6(-4-2y') = 5(-4+2y'),\quad -24-12y' = -20+10y',\quad -4 = 22y'$$

and so

$$y' = 2/11$$

The tangent line is

$$y - (-1) = \frac{2}{11}(x - (-2))$$

and so

$$y = \frac{2}{11}(x + 2) - 1$$

Problem 6. *(8 points)* Waterpark management decided to install a rectangular billboard with a picture of a dolphin under a popular waterslide. The end of the slide is three meters away from the wall where it starts (see picture). According to the manual, the slide has the shape of a parabola with equation $y = 2x^2$. Compute the area of the largest possible billboard that can fit into the space between the slide and the wall. *(For full credit, you need to justify somehow that what you found is really the maximum.)*

Solution: Let x be the coordinate of the lower left-hand (southwest) corner of the billboard. Then the area of the poster is

$$A(x) = (3 - x)2x^2 = 6x^2 - 2x^3$$

This is the function to be minimized, over the interval $[0, 3]$.

$$A'(x) = 12x - 6x^2 = 6x(2 - x)$$

So $A'(x) = 0$ *if* $x = 0$ *and if* $x = 2$. *Compare:*

$$A(0) = 0, \quad A(2) = 8, \quad A(3) = 0$$

Hence the maximum occurs at $x = 2$, *the maximum area of the billboard is* 8.

Problem 7. *(12 points)* You are driving a car along a highway at a steady 30 m/sec (which is 108 km/h) when you see an accident ahead. You slam on the brakes and deccelerate at a constant rate until the car stops. That is, your acceleration, measured in m/s^2, is given by the formula

$$a(t) = -9,$$

where t is the time measured in seconds from the moment you stepped on the breaks.

a. *(4 points)* Find the formula for the velocity $v(t)$ of the car, as a function of time t measured in seconds from the moment you stepped on the breaks.

Solution:

$$v(t) = \int a(t)\,dt, \quad v(t) = \int -9\,dt = -9t + C$$

To find C, use $v(0) = 30$. So $30 = -9 \cdot 0 + C$, so $C = 30$. Then

$$v(t) = 30 - 9t$$

b. *(4 points)* Find the formula for the distance $d(t)$ traveled by the car, as a function of time t measured in seconds from the moment you stepped on the breaks.

Solution:

$$d(t) = \int v(t)\,dt, \quad d(t) = \int 30 - 9t\,dt = 30t - \frac{9}{2}t^2 + C$$

To find C, use $d(0) = 0$, so $0 = 30 \cdot 0 - \frac{9}{2}0^2 + C$, so $C = 0$. Then

$$d(t) = 30t - \frac{9}{2}t^2$$

c. *(2 points)* Find the time t when the car stops.

Solution: car stops when $v(t) = 0$.

$$0 = 30 - 9t, \quad t = 30/9$$

d. *(2 points)* Find the distance it takes for the car to stop.

Solution:

$$d(30/9) = 30(30/9) - \frac{9}{2}(30/9)^2 = 30^2/18 = 100/2 = 50$$

Problem 8. *(8 points total)* Find the area enclosed between the curves $y = |x|$ and $y = 2 - x^2$.

Solution: $|x| = 2 - x^2$ *when* $x = -1$ *and when* $x = 1$ *(picture helps!). The function on top is* $2 - x^2$. *The funtion on bottom is* $|x$. *The area is*

$$\int_{-1}^{1} 2 - x^2 - |x|\, dx = \int_{-1}^{1} 2 - x^2\, dx - \int_{-1}^{1} |x|\, dx = (2x - \frac{1}{3}x^3)|_{-1}^{1} - 1 = 2 - \frac{1}{3} - (-2 + \frac{1}{3}) - 1 = 3 - \frac{2}{3} = 2\frac{1}{3}$$

To evaluate $\int_{-1}^{1} |x|\, dx$, *it is best to use geometry: two triangles, each of which has area* $1/2$.

Problem 9. *(8 points total)* A football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are b feet apart and that the hash mark line is a distance $a > 0$ from the right goal post. Find the distance d from the goal post line that gives the kicker the largest angle β.

Solution: for any $d > 0$,

$$\beta(d) = \arctan\left(\frac{a+b}{d}\right) - \arctan\left(\frac{a}{d}\right)$$

Then

$$\beta'(d) = \frac{1}{1 + \left(\frac{a+b}{d}\right)^2}\left(-\frac{a+b}{d^2}\right) - \frac{1}{1 + \left(\frac{a}{d}\right)^2}\left(-\frac{a}{d^2}\right) = -\frac{a+b}{d^2 + (a+b)^2} + \frac{a}{d^2 + a^2}$$

Set $\beta'(d)$ *equal to* 0, *get*

$$\frac{a+b}{d^2 + (a+b)^2} = \frac{a}{d^2 + a^2}, \quad (a+b)(d^2 + a^2) = a(d^2 + a^2 + b^2 + 2ab)$$

$$ad^2 + a^3 + bd^2 + ba^2 = ad^2 + a^3 + ab^2 + 2a^2 b, \quad bd^2 = ab^2 + a^2 b$$

The only critical point is

$$d = \sqrt{ab + a^2}$$

To justify that this only critical point is the maximum, one could say that when $d \to 0$, *the angle goes to* 0, *and when* $d \to \infty$, *the angle goes to* 0 *as well. Then the only critical point must be the maximum.*