Loyola University Chicago
Math 161, Fall 2010

Midterm 1

Name (print): ____________________ **Signature:** ____________________

Please do not start working until instructed to do so.

You have 75 minutes.

You must show your work to receive full credit.

No calculators.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Total. ________

Problem 1. *(40 points)* Find the following limits and derivatives. Put a $\boxed{\text{box around your final answer}}$.

a. *(5 points)* $\displaystyle\lim_{x\to 4}\frac{7x+2}{x^2-1}$

Solution: plug in $x=4$, get

$$\lim_{x\to 4}\frac{7x+2}{x^2-1}=\frac{28+2}{16-1}=2$$

b. *(5 points)* $\displaystyle\lim_{y\to\infty}\frac{10y-2y^2+5}{\sqrt{7+y^2+9y^4}}$

Solution: divide top and bottom by x^2, get

$$\lim_{y\to\infty}\frac{10y-2y^2+5}{\sqrt{7+y^2+9y^4}}=\lim_{y\to\infty}\frac{10/y-2+5/y^2}{\sqrt{7/y^4+1/y^2+9}}=\frac{-2}{\sqrt{9}}=-\frac{2}{3}$$

c. *(5 points)* $\displaystyle\lim_{x\to -2^+}\sqrt{\frac{x^2+3x+2}{x^2+x-2}}$

Solution: factor, simplify!

$$\lim_{x\to -2^+}\sqrt{\frac{x^2+3x+2}{x^2+x-2}}=\lim_{x\to -2^+}\sqrt{\frac{(x+1)(x+2)}{(x-1)(x+2)}}=\lim_{x\to -2^+}\sqrt{\frac{(x+1)}{(x-1)}}=\sqrt{\frac{-2+1}{-2-1}}=\sqrt{\frac{1}{3}}$$

d. *(5 points)* $\displaystyle\lim_{x\to 1^-}\frac{x^2+3x+2}{x^2+x-2}$

Solution:

$$\lim_{x\to 1^-}\frac{x^2+3x+2}{x^2+x-2}=\lim_{x\to 1^-}\frac{x+1}{x-1}=-\infty$$

because it is, essentially, $\dfrac{2}{0^-}$

e. *(5 points)* $\displaystyle\lim_{z\to 3^-}\frac{2z-6}{|z-3|}$

Solution: $z\to 3^-$ means $z<3$ so $z-3<0$ so $|z-3|=-(z-3)$, and so

$$\lim_{z\to 3^-}\frac{2z-6}{|z-3|}=\lim_{z\to 3^-}\frac{2z-6}{-(z-3)}=-2$$

f. *(5 points)* $\dfrac{d}{dx}\left(7x^5 - 11 + \ln(x) - \tan(x)\right)$

Solution:

$$\frac{d}{dx}\left(7x^5 - 11 + \ln(x) - \tan(x)\right) = 35x^4 + \frac{1}{x} - \frac{1}{\cos^2 x}$$

g. *(5 points)* $\dfrac{d}{dx}\left(\sqrt{5 + \cos(3^{4x})}\right)$

Solution:

$$\frac{d}{dx}\left(\sqrt{5 + \cos(3^{4x})}\right) = \frac{1}{2}\left(5 + \cos(3^{4x})\right)^{-\frac{1}{1}}\left(-\sin(3^{4x})\right)\ln 3\, 3^{4x}\, 4$$

h. *(5 points)* $\dfrac{d}{dx}\left(\dfrac{a^x + b}{\sin x + x^c}\right)$ where a, b, and c are constants

Solution:

$$\frac{d}{dx}\left(\frac{a^x + b}{\sin x + x^c}\right) = \frac{\ln a\, a^x\,(\sin x + x^c) - (a^x + b)(\cos x + cx^{c-1})}{(\sin x + x^c)^2}$$

Problem 2. *(10 points)* Use <u>the definition</u> of the derivative to find the derivative of $f(x) = \sqrt{5x^2 + 1}$.

Solution:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{5(x+h)^2 + 1} - \sqrt{5x^2 + 1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{5(x+h)^2 + 1} - \sqrt{5x^2 + 1}}{h}\,\frac{\sqrt{5(x+h)^2 + 1} + \sqrt{5x^2 + 1}}{\sqrt{5(x+h)^2 - 1} + \sqrt{5x^2 + 1}} \\
&= \lim_{h\to 0}\frac{5(x+h)^2 + 1 - 5x^2 - 1}{h(\sqrt{5(x+h)^2 + 1} + \sqrt{5x^2 + 1})} \\
&= \lim_{h\to 0}\frac{5x^2 + 10xh + 5h^2 - 5x^2}{h(\sqrt{5(x+h)^2 + 1} + \sqrt{5x^2 + 1})} \\
&= \lim_{h\to 0}\frac{10x + 5h}{\sqrt{5(x+h)^2 + 1} + \sqrt{5x^2 + 1}} \\
&= \frac{10x}{2\sqrt{5x^2 + 1}}
\end{aligned}$$

Problem 3. *(15 points)* Consider the function

$$f(x) = \begin{cases} x^2 & \text{if} \quad x < 1 \\ 2x - 1 & \text{if} \quad 1 \le x \le 3 \\ 5 & \text{if} \quad x > 5 \end{cases}$$

a. *(5 points)* Is this function continuous at $x = 1$? Is this function continuous at $x = 3$. Justify your answer.

Solution: at $x = 1$ $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$, $f(1) = 1$, $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} 2x - 1 = 1$, *so the function is continuous. At* $x = 3$ $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2x - 1 = 5$, $f(3) = 5$, *so the function is continuous. (The domain of* $f(x)$ *does not extend just to the right of* 3*, so there is no* $\lim_{x \to 3^+} f(x)$ *to consider.)*

b. *(5 points)* Is this function differentiable at $x = 1$? Is this function differentiable at $x = 3$. Justify your answer.

Solution: at $x = 1$*, the function is differentiable because* $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$ *exists:*

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 - 1}{h} = 1, \quad \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{2(1+h) - 1 - 1}{h} = 1$$

Essentially, the slope to the left and the slope to the right are equal. At $x = 3$*, the function is differentiable from the left.*

NOTE: if you said that f *is not continuous at* $x = 3$*, then it is correct to conclude* f *is not differentiable at* $x = 3$*.*

c. *(5 points)* With $f(x)$ as above and $g(x)$ such that $g(1) = 2$ and $g'(1) = -4$, find $h'(1)$, where $h(x) = f(g(x))$.

Solution: $h'(x) = f'(g(x))g'(x)$ *by chain rule, so* $h'(1) = f'(g(1))g'(1) = f'(2)(-4) = 2(-4) = -8$

NOTE: I intended for the problem to say that $f(x) = 5$ *for* $x > 3$ *(not for* $x > 5$*). Then,* f *is continuous at both* 1 *and* 3*, and* f *is differentiable at* 1 *but not at* 3*.*

Problem 4. *(15 points)* Consider the curve

$$x^2 - 4xy + y^4 = 0.$$

a. *(5 points)* Find the general formula for $\frac{dy}{dx}$, the slope of the curve, in terms of x and y.

Solution: implicit differentiation:

$$2x - 4y - 4xy' + 4y^3y' = 0$$

Solve for y', get

$$\frac{dy}{dx} = \frac{4y - 2x}{4y^3 - 4x}$$

b. *(5 points)* Find the equation of the line tangent to the curve at the point $\left(\frac{1+\sqrt{3}}{2}, 1\right)$.

Solution: when $x = \frac{1+\sqrt{3}}{2}$, $y = 1$, the slope is

$$m = \frac{4y - 2x}{4y^3 - 4x} = \frac{4 - 1 - \sqrt{3}}{4 - 2 - 2\sqrt{3}} = \frac{3 - \sqrt{3}}{2 - 2\sqrt{3}}$$

The tangent line is

$$y - 1 = \frac{3 - \sqrt{3}}{2 - 2\sqrt{3}}\left(x - \frac{1 + \sqrt{3}}{2}\right)$$

c. *(5 points)* Find the point (or points) on the curve where the tangent line is horizontal.

Solution: tangent line is horizontal where $\frac{dy}{dx} = 0$, so set $y' = 0$ in $2x - 4y - 4xy' + 4y^3y' = 0$. Get $2x - 4y = 0$, so $x = 2y$. Now, find points on the curve $x^2 - 4xy + y^4 = 0$ where $x = 2y$:

$$(2y)^2 - 4(2y)y + y^4 = 0, \quad 4y^2 - 8y^2 + y^4 = 0, \quad y^4 - 4y^2 = 0, \quad y^2(y^2 - 4) = 0$$

and so $y = 0$ or $y = 2$ or $y = -2$. The points on the curve are $(0, 0)$, $(4, 2)$, $(-4, -2)$. There is a problem with $(0, 0)$ though — it is not possible to tell what the slope at $(0, 0)$ is. So, the answer is this: the tangent line is horizontal at $(4, 2)$ and $(-4, -2)$, and I am not quite sure about $(0, 0)$.

Problem 5. *(10 points total)* When a bactericide was added to a nuntrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population, measured in thousands, at time $t > 0$ (hours) is

$$b(t) = te^{-\frac{t}{3}}.$$

a. *(5 points)* How fast is the bacterium population changing at time t?

Solution: this is a question about the rate of change of $b(t)$, so about the derivative of $b(t)$

$$b'(t) = \left(te^{-\frac{t}{3}}\right)' = e^{-\frac{t}{3}} - te^{-\frac{t}{3}}\frac{1}{3} = e^{-\frac{t}{3}}\left(1 - \frac{t}{3}\right)$$

b. *(5 points)* When does the population stop growing?

Solution: set $b'(t) = 0$, get

$$0 = e^{-\frac{t}{3}}\left(1 - \frac{t}{3}\right)$$

so

$$0 = 1 - \frac{t}{3}$$

so $t = 3$.

c. *(5 points)* What is the maximum size of the population?

Solution: maximum size happens when population stops growing, so

$$b(3) = 3e^{-1}$$