Loyola University Chicago
Math 161, Fall 2010

Midterm 2 Sample

Name (print): ____________________ **Signature:** ____________________

Please do not start working until instructed to do so.

You have 75 minutes.

You must show your work to receive full credit.

No calculators.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Total. __________

Problem 1. *(20 points)* Find the following limits and integrals. Put a $\boxed{\text{box around your final answer}}$.

a. *(4 points)* $\displaystyle\lim_{x\to\infty} \frac{\ln(2x+4)}{7x+11}$

Solution: this is an indeterminate form of the $\frac{\infty}{\infty}$ kind, use L'Hopital rule:

$$\lim_{x\to\infty} \frac{\ln(2x+4)}{7x+11} = \lim_{x\to\infty} \frac{\frac{2}{2x+4}}{7} = \lim_{x\to\infty} \frac{2}{7(2x+4)} = 0$$

b. *(4 points)* $\displaystyle\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{3x}$

Solution: recall that $\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^{y} = e$ *and use it below:*

$$\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{3x} = \lim_{x\to\infty} \left(1+\frac{1}{\frac{x}{4}}\right)^{3x} = \lim_{x\to\infty} \left(1+\frac{1}{\frac{x}{4}}\right)^{12\frac{x}{4}} = \left(\lim_{x\to\infty} \left(1+\frac{1}{\frac{x}{4}}\right)^{\frac{x}{4}}\right)^{12} = e^{12}$$

Another way to solve this is to do this

$$\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{3x} = e^{\ln \lim_{x\to\infty}\left(1+\frac{4}{x}\right)^{3x}} = e^{\lim_{x\to\infty} \ln\left(1+\frac{4}{x}\right)^{3x}} = e^{\lim_{x\to\infty} 3x\ln\left(1+\frac{4}{x}\right)} = e^{\lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{3x}}}$$

and then use L'Hopital's rule.

c. *(4 points)* $\displaystyle\lim_{x\to 0} \frac{\tan(\pi x)}{\ln(1+x)}$

Solution:

$$\lim_{x\to 0} \frac{\tan(\pi x)}{\ln(1+x)} = \lim_{x\to 0} \frac{\frac{\pi}{\cos^2(\pi x)}}{\frac{1}{1+x}} = \pi$$

d. *(4 points)* $\displaystyle\int 3x^5 - 2x + 1\,dx$

Solution:

$$\int 3x^5 - 2x + 1\,dx = \frac{1}{2}x^6 - x^2 + x + C$$

e. *(4 points)* $\displaystyle\int e^{-4x} + \frac{3}{1+x^2}\,dx$

Solution:

$$\int e^{-4x} + \frac{3}{1+x^2}\,dx = -\frac{1}{4}e^{-4x} + 3\tan^{-1}(x) + C$$

Problem 2. *(10 points)* Use Newton's method to approximate the value of $\sqrt{12}$. Pick the initial approximation x_1 reasonably. Then find the second approximation x_2 and the third approximation x_3. You do not need to simplify the second approximation.

Solution: we need to find x which equals $\sqrt{12}$. Hence $x = \sqrt{12}$, $x^2 = 12$, $x^2 - 12 = 0$, so in other words, we need to find a root of $f(x) = 0$ where $f(x) = x^2 - 12$. A reasonable initial guess is $x_0 = 3$, another reasonable guess is $x_0 = 4$. We have $f'(x) = 2x$ and then

$$x_1 = x_0 - \frac{f(x_0)}{f(x_0)} = 3 - \frac{3^2 - 12}{2 \cdot 3} = 3\frac{1}{2}$$

$$x_2 = x_1 - \frac{f(x_1)}{f(x_1)} = 3 - \frac{\left(3\frac{1}{2}\right)^2 - 12}{2 \cdot \left(3\frac{1}{2}\right)}$$

Problem 3. *(10 points)* Find horizontal and vertical asymptotes (if any exist), critical points (if any exist), and classify the critical points as local/global minima/maxima/neither, for the function

$$f(x) = \frac{e^x}{1 + x^2}$$

Solution: No vertical asymptotes because the denominator is always greater or equal than 1 (so positive, never 0). To look for horizontal asymptotes:

$$\lim_{x \to -\infty} \frac{e^x}{1 + x^2} = \frac{0}{\infty} = 0, \qquad \lim_{x \to \infty} \frac{e^x}{1 + x^2} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{e^x}{2} = \infty$$

Hence $y = 0$ is a horizontal asymptote. To find critical points:

$$f'(x) = \frac{e^x(1 + x^2) - e^x 2x}{(1 + x^2)^2} = \frac{e^x (x - 1)^2}{(1 + x^2)^2}$$

and $f'(x) = 0$ when $x = 1$. A closer look at the derivative shows that $f'(x) \geq 0$ for all x, and so f is nondecreasing. Hence $x = 1$ is not a local/global min/max. It is just a boring critical point.

Problem 4. *(10 points)* You are designing a rectangular poster to contain 50 square inches of printing with a 4 inch margin at the top and bottom and a 2 inch margin at each side. What overall dimensions will minimize the amount of paper used?

Solution: Let x be the height of the whole poster, let y be the width of the whole poster. We need to minimize $A = xy$. Printed area being 50 means that $(x-8)(y-4) = 50$, so $y = 4 + \frac{50}{x-8}$. Then

$$A(x) = x\left(4 + \frac{50}{x-8}\right)$$

Need to minimize this function over $x > 8$.

$$A'(x) = 4 - \frac{400}{(x-8)^2}, \qquad \textit{so } A'(x) = 0 \textit{ gives } x = 18.$$

Is this really a minimum?

$$A''(x) = \frac{800}{(x-8)^3} > 0$$

so the function is concave up, so $x = 18$ is the absolute minimum. When $x = 18$, $y = 9$.

Problem 5. *(10 points)* Suppose that an ostrich 5 ft tall is walking at a speed of 4 ft/sec directly towards a light 10 ft high. How fast is the tip of the ostrich's shadow moving along the ground?

Solution: Let x be the distance of ostrich from light. We know $\frac{dx}{dt} = -4$. Let y be the distance of the tip of the shadow from light. We need $\frac{dy}{dy}$. Similar triangles say

$$\frac{y-x}{5} = \frac{y}{10}$$

and so $y = 2x$. Then

$$\frac{dy}{dt} = 2\frac{dx}{dt} = 2\,4 = 8$$