Loyola University Chicago
Math 201, Fall 2009

Final Exam

Name (print): ______________________ **Signature:** ______________

Please do not start working until instructed to do so.

You have 2 hours.

You must show your work to receive full credit.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. __________

Problem 2. __________

Problem 3. __________

Problem 4. __________

Problem 5. __________

Problem 6. __________

Problem 7. __________

Problem 8. __________

Problem 9. __________

Total. __________

Problem 1. *(10 points total)* Let P, Q, and R be statements. Is $(P \text{ AND } Q) \implies R$ equivalent to $(P \implies R) \text{ OR } (Q \implies R)$?

Answer / solution: Yes, they are equivalent. Check using truth table.

Problem 2. *(8 points total)* Multiply $(102)_3$ and $(20112)_3$ in base 3.

Answer / solution: $(2122201)_3$

Problem 3. *(14 points total)*

a. *(7 points)* Find all integers x such that $x^3 + 4x + 1 \equiv 0 \pmod 5$.

Answer / solution: use brute force, get $x \equiv 3 \pmod 5$ *or* $x = 3 + 5k$, $k \in \mathbb{Z}$.

b. *(7 points)* Find all nonnegative integers such that $x^3 + 4x + 1 \equiv 0 \pmod 5$ and $3x \equiv 2 \pmod 7$.

Answer / solution: Put $x = 3 + 5k$ *into* $3x \equiv 2 \pmod 7$, *get* $9 + 15k \equiv 2 \pmod 7$, *simplify to* $k \equiv 0 \pmod 7$. *So* $k = 7l$, $l \in \mathbb{Z}$, *and* $x = 3 + 5k = 3 + 35l$, $l \in \mathbb{Z}$. *But we need nonnegative* x, *so the final answer is* $x = 3 + 35l$, $l \in \mathbb{Z}$, $l \geq 0$.

Problem 4. *(10 points total)* Suppose that $ac \equiv bc \pmod{m}$ and $\gcd(c, m) = 1$. Show that $a \equiv b \pmod{m}$.

Answer / solution: $ac \equiv bc$ *(mod* m*) means that* $m|ac - bc$ *so* $m|c(a-b)$*. Since* $gcd(c, m) = 1$*,* c *and* m *have no common prime factors. Every prime factor of* m *divides* $c(a - b)$*. Hence, every prime factor of* m *divides* $a - b$*. So then* m *divides* $a - b$*, so* $a \equiv b$ *(mod* m*).*

Problem 5. *(10 points total)* Let $x_0 = 5$, $x_1 = 1$, and $x_n = -5x_{n-1} + 14x_{n-2}$ for $n = 2, 3, \ldots$. Show that $x_n = 4 \cdot 2^n + (-7)^n$ for all $n = 0, 1, 2, \ldots$.

Answer / solution: use strong induction.

Problem 6. *(16 points)*

a. *(6 points)* Find $\gcd(990, 714)$.

Answer / solution: 6

b. *(5 points)* Find all integers x and y such that $990x + 714y = 18$.

Answer / solution:

$$x = 13 - 119k, \quad y = -18 + 165k, \quad k \in \mathbb{Z}.$$

c. *(5 points)* Find all integers x and y such that $990x + 714y = 25$.

Answer / solution: No solutions because 6 *does not divide* 25.

Problem 7. *(10 points total)* Prove that $2^{70} + 3^{70}$ is divisible by 13.

Answer / solution: Use brute force or Litte Fermat's Theorem to get that $2^12 \equiv 1$ *(mod 13). So* $2^{70} = \left(2^{12}\right)^5 \cdot 2^{10} \equiv 2^{10}$ *(mod 13). Also, get* $3^3 \equiv 1$ *(mod 13) so* $3^{70} \equiv \left(3^3\right)^{23} \cdot 3 \equiv 3$ *(mod 13). So*

$$2^{70} + 3^{70} \equiv 2^{10} + 3 = 1024 + 3 = 1027 = 79 \cdot 13$$

and hence $13|2^{70} + 3^{70}$. *Alternatively, you can rely on Little Fermat's Theorem to get that* $2^{12} \equiv 1$ *(mod 13), etc...*

Alternative solution: use the identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1}.$$

Then

$$2^{70} + 3^{70} = 4^{35} - (-9)^{35} = (4 - (-9))\,(\text{ some integer }) = 13\,(\text{ some integer })$$

Problem 8. *(12 points total)* How many elements does each of the following sets have? *Frame your answer. No partial credit.*

a. *(4 points)* Nonnegative divisors of 3969.

Answer / solution: $5 \cdot 3 = 15$.

b. *(4 points)* Nonnegative integers, greater than 100 and less than 660, that can be written with distinct digits.

Answer / solution: count the numbers between 100 *and* 599*, then count the numbers between* 600 *and* 660*, get* $5 \cdot 9 \cdot 8 + 6 \cdot 8$.

c. *(4 points)* Ways of ordering the elements of the set $\{A, B, C, D, E, F, G, H, I, J\}$ in a line such that either A or B is first and G is last.

Answer / solution: $2 \cdot 8!$

Problem 9. *(10 points)* Prove using mathematical induction that there is $n!$ permutations of a set with n elements. *Only partial credit will be given for proofs that do not use induction. Recall that a permutation of a set with n elements is, essentially, an arrangements of n elements in a line.*

Answer / solution: The key step is showing this: if there are $n!$ *ways of arranging* n *elements in line, then there are* $(n+1)!$ *ways of arranging* $n+1$ *elements in line. There are ways to thos the latter directly, but that is not induction!*

Take a set with $n+1$ *elements, say* $\{a_1, a_2, \ldots, a_{n+1}\}$*. Take the first* n *elements* $\{a_1, a_2, \ldots, a_n\}$ *and arrange them in a line. There is* $n!$ *ways to do this. Then, there are also* $n+1$ *different spots to put the element* a_{n+1}*: it could be first, second, third,...,* $n+1$*-st. Hence, there is* $(n+1)n!$ *ways of arranging* $n+1$ *elements in a line, and* $(n+1)n! = (n+1)!$.