Loyola University Chicago
Math 201, Spring 2010

Final Exam

Name (print): ______________________ **Signature:** ______________

Please do not start working until instructed to do so.

You have 2 hours.

You must show your work to receive full credit.

You may use one double-sided 8.5 by 11 sheet of handwritten (by you) notes.

Problem 1. ________

Problem 2. ________

Problem 3. ________

Problem 4. ________

Problem 5. ________

Problem 6. ________

Problem 7. ________

Problem 8. ________

Problem 9. ________

Total. ________

Problem 1. *(16 points total)* For each of the true/false questions below, circle the right answer. For each of the other questions, write a numerical answer on the line. No partial credit.

a. *(2 points)* $\forall x \in \mathbb{Z}\ \exists y \in \mathbb{Z}$ such that $x + y \geq 0$.

[True.] **False.**

b. *(2 points)* For any prime number p and any integer a, $a^{p-1} \equiv 1 \pmod{p}$.

True. **[False.]**

c. *(2 points)* For any statements A and B, $(A \implies B)\ OR\ ((NOT\ A) \implies B)$ is true.

[True.] **False.**

d. *(2 points)* $1.23456567567567567567... = 1.23456\dot{5}\dot{6}\dot{7}$ is irrational.

True. **[False.]**

e. *(2 points)* Ways of arranging all of the elements of the set $\{A, B, C, D, E, F\}$ in a line such that A is either in the first place or in the last place.

Answer: $2 \cdot 5! = 240$

f. *(2 points)* $7920 = 16 \cdot 9 \cdot 5 \cdot 11$. How many different nonnegative divisors does 7920 have?

Answer: $5 \cdot 3 \cdot 2 \cdot 2 = 60$

g. *(2 points)* Convert $(1011011)_2$, which is written in base 2, to base 10.

Answer: $1 + 2 + 8 + 16 + 64 = 91$

h. *(2 points)* Suppose that p and q are prime numbers with $p \neq q$. Then $x \equiv 7 \pmod{pq}$ is equivalent to $x \equiv 7 \pmod{p}$ and $x \equiv 7 \pmod{q}$.

[True.] **False.**

Problem 2. *(9 points total)* Let $a = 98910012$ and $b = 9591000110$.

a. *(3 points)* Is ab divisible by 8? Why?

Solution: $8 = 2^3$, *so check divisibility of* a *and* b *by powers of* 2. *Recall the test for divisibility by powers of* 2: 2^p *divides a number* x *if* 2^p *divides the number made up of the last* p *digits of* x. *Now,* $4|a$ *because* $4|12$, *and* $2|b$ *because* $2|0$. *Hence* $4 \cdot 2|ab$. *Answer is YES.*

b. *(3 points)* Is ab divisible by 16? Why?

Solution: we already know $4|a$ *and* $2|b$. *Now note* $8 \not| a$ *because* $8 \not| 12$, *and* $4 \not| b$ *because* $4 \not| 110$. *Hence* 8 *is the highest power of* 2 *that divides* ab. *Answer is NO.*

c. *(3 points)* Is ab divisible by 15? Why?

Solution: $15 = 3 \cdot 5$. $3|a$ *because* $3|9 + 8 + 9 + 1 + 1 + 2 = 30$, $5|b$ *because the last digit of* b *is* 0 *or* 5. *Hence* $15|ab$. *Answer is YES.*

Problem 3. *(8 points total)* Let $p, q, x, y \in \mathbb{Z}$ and consider the statement:

If $p|x$ and $q|y$ then $pq|xy$.

a. *(4 points)* Write the converse statement and determine if it is true or not.

Solution: converse statement:

If $pq|xy$ *then* $p|x$ *and* $q|y$.

It is FALSE. For example, $2 \cdot 2|4 \cdot 1$ *but it is not true that* $2|4$ *and* $2|1$.

b. *(4 points)* Write the contrapositive statement and determine if it is true or not.

Solution: contrapositive statement is "If NOT $pq|xy$ *then NOT* $(p|x$ *and* $q|y)$*" which simplifies to*

If $pq \not| xy$ *then* $p \not| x$ *or* $q \not| y$.

It is TRUE because the original statement is true.

Problem 4. *(10 points)* Find the greatest common divisor of 4522 and 434.

Solution: EEA looks like this:

1	0	4522
0	1	434
1	−10	182
−2	21	70
5	−52	42
−7	73	28
12	−125	14
−31	323	0

Hence $gcd(4522, 434) = 14$.

Problem 5. *(10 points total)* Prove that for all natural numbers n,

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Solution:

Problem 6. *(15 points total)* The Extended Euclidean Algorithm applied to 1001 and 126 looks like this:

$$\begin{array}{rrl} 1 & 0 & 1001 \\ 0 & 1 & 126 \\ 1 & -7 & 119 \\ -1 & 8 & 7 \\ 18 & -143 & 0 \end{array}$$

and consequently, $\gcd(1001, 126) = 7$. Using this information, answer the questions below:

a. *(3 points)* Find all integer solutions to $1001x + 126y = 7$.

Solution: next to last line in EEA gives $1001(-1)+126(8) = 7$. *Last line in EEA gives* $1001(18)+126(-143) = 0$ *so also* $1001(18k) + 126(-143k) = 0$ *for all* $k \in \mathbb{Z}$. *Hence*

$$1001(-1 + 18k) + 126(8 - 143k) = 7$$

and the complete solution is $x = -1 + 18k$, $y = 8 - 143k$, $k \in \mathbb{Z}$.

b. *(3 points)* Find all integer solutions to $1001x + 126y = 21$.

Solution: next to last line in EEA gives $1001(-1) + 126(8) = 7$ *so also* $1001(-3) + 126(24) = 21$. *Last line in EEA gives* $1001(18) + 126(-143) = 0$ *so also* $1001(18k) + 126(-143k) = 0$ *for all* $k \in \mathbb{Z}$. *Hence*

$$1001(-3 + 18k) + 126(24 - 143k) = 21$$

and the complete solution is $x = -3 + 18k$, $y = 24 - 143k$, $k \in \mathbb{Z}$.

c. *(3 points)* Find all integer solutions to $1001x + 126y = 15$.

Solution: gcd$(1001, 126) = 7$, 7 *does not divide* 15, *so there are NO solutions.*

d. *(3 points)* Find all nonnegative integer solutions to $1001x + 126y = 126$.

Solution: note that $126/7 = 18$, *and just like in part a above, get*

$$1001(-18 + 18k) + 126(144 - 143k) = 126.$$

Now we need $-18+18k \geq 0$, *which simplifies to* $k \geq 1$, *as well as* $144-143k \geq 0$, *which simplifies to* $144/143 \geq k$. *The only* k *satisfying both inequalities is* $k = 1$. *Hence, the only nonnegative solution is* $x = -18 + 18 = 0$, $y = 144 - 143 = 1$.

e. *(3 points)* Find all integer solutions to $126z \equiv 1008 \pmod{1001}$.

Solution: $126z \equiv 1008$ *(mod* 1001*) is equivalent to* $126z \equiv 7$ *(mod* 1001*) which is equivalent to existence of* $x \in \mathbb{Z}$ *such that* $126z + 1001x = 7$. *This was solved in a above, so* $z = 8 - 143k$, $k \in \mathbb{Z}$.

Problem 6. *(10 points)* State Fermat's Little Theorem and use it to prove the following:

- For any prime number p and any integer a, $a^p - a$ is divisible by p.

Solution: FLT says this: if $p \in \mathbb{Z}$ is prime and $a \in \mathbb{Z}$ is not divisible by p, then $a^{p-1} \equiv 1$ (mod p).

We need to prove that for any prime number p and any integer a, $a^p \equiv a$ (mod p). If $p \nmid a$, then FLT implies $a^{p-1} \equiv 1$ (mod p), multiplying both sides by a gives $a^p \equiv a$ (mod p). If $p|a$, then $a \equiv 0$ (mod p), hence $a^p \equiv 0$ (mod p), and $a^p \equiv a$ (mod p) because $0 \equiv 0$ (mod p).

Problem 7. *(10 points total)* Solve the simultaneous congruences:

$$3x \equiv 4 \pmod 5, \qquad 4x \equiv 5 \pmod 9.$$

Solution: the inverse of $[3]$ in $\mathbb{Z}_5$ is $[2]$, because $[2][3] = [6] = [1]$. Multiply both sides of first congruence by 2, get $6x \equiv 8$, so $x \equiv 3$ (mod 5). Hence $x = 3 + 5k$. Plug this into second congruence, and simplify: $4(3 + 5k) \equiv 5$, $2k \equiv 2$ (mod 9). Then $k \equiv 1$ (mod 9) and so $k = 1 + 9l$, $l \in \mathbb{Z}$. Hence $x = 3 + 5(1 + 9l) = 8 + 45l$.

Problem 8. *(8 points total)* The *Fibonacci numbers* are defined as follows:

$$a_1 = 1, \quad a_2 = 1, \quad a_n = a_{n-2} + a_{n-1} \ \text{ for } n \geq 3.$$

For example, $a_3 = a_{3-2} + a_{3-1} = a_1 + a_2 = 2$, $a_4 = a_2 + a_3 = 3$, $a_5 = 5$, $a_6 = 8$, etc. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n} a_r^2 = a_n a_{n+1}.$$

Solution: use induction. First, for $n = 1$, get $\sum_{k=1}^{n} a_r^2 = \sum_{k=1}^{1} a_r^2 = a_1^2 = 1^2 = 1$, while $a_n a_{n+1} = a_1 a_2 = 1 \cdot 1 = 1$, and $1 = 1$. Now suppose that $\sum_{k=1}^{n} a_r^2 = a_n a_{n+1}$ and try to show that $\sum_{k=1}^{n+1} a_r^2 = a_{n+1} a_{n+2}$. We have

$$\sum_{k=1}^{n+1} a_r^2 = \left(\sum_{k=1}^{n} a_r^2\right) + a_{n+1}^2 = a_n a_{n+1} + a_{n+1}^2 = (a_n + a_{n+1})\, a_{n+1} = a_{n+2} a_{n+1},$$

where the second equality above comes from the inductive assumption and the last equality comes from the definition of Fibonnaci numbers. We are done.

Problem 9. *(8 points)* Let m be a positive integer. Prove that if $2^m - 1$ is prime then m is prime.

This may be useful: $a^k - b^k = (a - b)\left(a^{k-1} + a^{k-2}b + a^{k-3}b^2 + \cdots + ab^{k-2} + b^{k-1}\right)$.

Solution: let's prove the contrapositive statement:

If m is not prime then $2^m - 1$ is not prime.

If m is not prime, then $m = pq$ for some $p, q \in \mathbb{Z}$, $1 < p, q < m$. Then, using the hint,

$$7^m - 1 = 2^{pq} - 1 = (2^p)^q - 1^q = (2^p - 1)\left((2^p)^{q-1} + (2^p)^{q-2}\, 1 + \cdots + (2^p)\, 1^{q-2} + 1^{q-1}\right).$$

In particular, $2^p - 1 | 2^m - 1$, and because $1 < 2^p - 1 < 2^m - 1$, this means that $2^m - 1$ is not prime. The contrapositive is thus proven. The original statement is equivalent to the contrapositive, so it is proved as well.