Loyola University Chicago
Math 201, Spring 2010

Sample Quiz 2 Solutions

Name (print): ______________________ **Signature:** ______________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. (5 pts) Prove that, given three integers a, b, and c, if $a|b$ and $b|c$, then $a|c$.

Solution: $a|b$ *means that there exists an integer* m *such that* $b = ma$. $b|c$ *means that there exists an integer* n *such that* $c = nb$. *Hence* $c = nb = n(ma) = (nm)a$. *Since* nm *is an integer,* $c = (nm)a$ *means that* $a|c$.

Problem 2. (5 pts) Find all pairs of nonnegative integers x and y such that $509x - 132y = 2$.

Solution: Extended Euclidean Algorithm applied to $509x + 132y = 2$ *yields*

1	0	509
0	1	132
1	−3	113
−1	4	19
6	−23	18
−7	27	1
132	−509	0

Next to last row says $509(-7)+132(27) = 1$, *hence* $509(-14)+132(54) = 2$. *Last row says* $509(132)+132(-509) = 0$, *hence* $509(132k) + 132(-509k) = 0$ *for all* $k \in \mathbb{Z}$. *Adding yields* $509(-14 + 132k) + 132(54-509k) = 2$, *hence* $509(-14+132k)-132(-54+509k) = 2$. *Thus, all solutions to* $509x-132y = 2$ *are given by* $x = -14 + 132k$, $y = -54 + 509k$. *We need* $x = -14 + 132k \geq 0$, $y = -54 + 509k \geq 0$, *so* $k \geq 14/132$, $k \geq 54/509$, *which amounts to* $k > 0$ *(or, equivalently,* $k \geq 1$*). Hence, all nonnegative solutions are given by*

$$x = -14 + 138k,\ y = -54 + 509k, \quad k \geq 1.$$

Problem 3. (5 pts) Your phone company charges 15 cents per minute for long distance calls within US and 39 cents per minute for calls to Europe. You receive a monthly bill for long distance calls shows \$16.70. explain why this bill must be incorrect.

Solution: Let x be the number of calls within US and y the number of calls to Europe. Then

$$15x + 39y = 1670$$

Is this equation possible with x and y integer? No, because $gcd(15, 39) = 3$ and 3 does not divide 1670.

Problem 4*. (5 pts) Let a and b be nonzero integers. Show that $\gcd(a, b) = 1$ if and only if $\gcd(a, a + b) = 1$.

Solution: First, we show that if $gcd(a, b) = 1$ then $gcd(a, a + b) = 1$. Obviously, $1|a$ and $1|a + b$. Suppose that d is a common divisor of a and $a + b$. Then $d|a$, $d|a + b$, and so $d|a + b - a$, in other words, $d|b$. So d is a common divisor of a and b, and so $d \leq 1$. Consequently, $gcd(a, a + b) = 1$.

Second, we show that if $gcd(a, a + b) = 1$ then $gcd(a, b) = 1$. If $gcd(a, a + b) = 1$ then there exist integer x, y such that $ax + (a + b)y = 1$. The last equationc can be changed to $a(x + y) + by = 1$. Since 1 is a common divisor of a and b, the equation $a(x + y) + by = 1$ implies that 1 is the greatest common divisor of a and b.