Loyola University Chicago
Math 201, Section 001, Fall 2009

Quiz 5 Sample

Name (print): ______________________________ **Signature:** ____________________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. (5 pts) Prove, by induction, that for every $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}.$$

Problem 2. (5 pts) Prove: if $1 \leq r \leq n$ then

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}.$$

Solution: see textbook!

Problem 3. (5 pts) Prove, by strong induction, that if

$$a_0 = 5,\ a_1 = -10, \quad a_n = -4(a_{n-1} + a_{n-2}) \text{ for } n = 2, 3, \ldots$$

then $a_n = 5(-2)^n$.

Solution: we need to use strong induction. First, check $n = 0$ and $n = 1$: $x_0 = 5(-2)^0 = 5$, $x_1 = 5(-2)^1 = -10$. Now, suppose $x_k = 5(-2)^k$ for $n = 0, 1, 2, \ldots, n$. We will show that $x_{n+1} = 5(-2)^{k+1}$. Using the inductive assumption (we only need it for $k = n$ and $k = n-1$), we have

$$x_{n+1} \quad = \quad -4(a_n + a_{n-1} = -4(5(-2)^{k+1} + 5(-2)^k) = -20\,(-2)^k(-2+1) = 20(-2)^k = 5(-2)^{k+2}$$

We are done!

Problem 4. (5 pts) Prove that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$.

Solution: use induction. For $n = 1$, $5^n - 4n - 1 = 0$, and $16|0$. Suppose that $5^n - 4n - 1$ is divisible by 16. In other words, $5^n - 4n - 1 = 16k$ for some $k \in \mathbb{Z}$. Need to show that $5^{n+1} - 4(n+1) - 1$ is divisible by 16.

$$5^{n+1} - 4(n+1) - 1 = 5(16k + 4n + 1) - 4n - 5 = 16(5k + n)$$

Done!

Problem 5. (5 pts) Prove that the product of r consecutive positive integers is divisible by $r!$.

Solution: need to prove that $r!$ divides $(n+1)(n+2)\ldots(n+r)$ for any $n \in \mathbb{N}$. Recall that we know that $\binom{p}{r}$ is integer for all $p \geq r$. Try $p = n + r$ then. Get

$$\binom{n+r}{r} = \frac{(n+1)(n+2)\ldots(n+r)}{r!}.$$

Done!