Loyola University Chicago **Quiz 5**
Math 201, Section 001, Fall 2009

Name (print): ______________________ **Signature:** ______________________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. (5 pts) Prove, by induction, that for every $n \in \mathbb{N}$,

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Solution:
First step: for $n = 1$, *left-hand side is* 1^2, *right-hand side is* $\frac{1(1+1)(2\cdot 1+1)}{6} = 1$, *so they are equal.*
Second step: suppose that $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ *and shown that* $1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{(n+1)(n+2)(2(n+1)+1)}{6}$. *Using the supposition, we get that*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

Now, do algebra:

$$\begin{aligned} \frac{n(n+1)(2n+1)}{6} + (n+1)^2 &= (n+1)\left[\frac{n(2n+1)}{6} + (n+1)\right] = (n+1)\frac{2n^2+n+6n+6}{6} \\ &= (n+1)\frac{2n^2+7n+6}{6} = \frac{(n+1)(n+2)(2n+3)}{6} \end{aligned}$$

Done!

Problem 2. (5 pts) Prove, by induction, that for every $n \in \mathbb{N}$ greater than 1,

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 3 - \frac{2}{(n+1)!}.$$

Solution:
First step: for $n = 2$, *left-hand side is* $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} = 1 + 1 + \frac{1}{2} = 2\frac{1}{2}$, *right-hand side is* $3 - \frac{2}{3!} = 3 - \frac{1}{3} = 2\frac{2}{3}$. *We have* $2\frac{1}{2} < 2\frac{2}{3}$.
Second step: suppose that $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < 3 - \frac{2}{(n+1)!}$ *and show that* $\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} < 3 - \frac{2}{(n+2)!}$. *Using the supposition, we get that*

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{(n+1)!} < 3 - \frac{2}{(n+1)!} + \frac{1}{(n+1)!}.$$

It is left to show that

$$3 - \frac{2}{(n+1)!} + \frac{1}{(n+1)!} \le 3 - \frac{2}{(n+2)!},$$

which simplifies to

$$\frac{1}{(n+1)!} + \frac{2}{(n+2)!} \le \frac{2}{(n+1)!}.$$

Multiply both sides by $(n+2)!$, *get* $n + 2 + 2 \le 2(n+2)$, *which is true. Done!*

Problem 3. (6 pts) Let $a_0 = 0$, $a_1 = 1$, and for $n \geq 2$, let $a_n = 2a_{n-1} - a_{n-2} + 2$.

(a) Calculate a_2, a_3, a_4, and a_5.

(b) Predict a formula for a_n based on your calculations.

(c) Prove your prediction from (b).

Solution: $a_0 = 0$, $a_1 = 1$, $a_2 = 2a_1 - a_0 + 2 = 4$, $a_3 = 2a_2 - a_1 + 2 = 9$, $a_4 = 2a_3 - a_2 + 2 = 16$, $a_5 = 2a_4 - a_3 + 2 = 25$. *Guess that* $a_n = n^2$. *Now prove it, using strong induction. Check* a_0 *and* a_1 *directly, and yes,* $a_0 = 0^2$, $a_1 = 1^2$. *Now, suppose that for* $k = 0, 1, 2, \ldots, n$, *we have* $a_k = k^2$. *Need to prove that* $a_{n+1} = (n+1)^2$.

$$a_{n+1} = 2a_n - a_{n-1} + 2 = 2n^2 - (n-1)^2 + 2 = 2n^2 - n^2 + 2n - 1 + 2 = n^2 + 2n + 1 = (n+1)^2.$$

Done!

Problem 4. (4 pts) State the Binomial Theorem and use it to prove that, for all $n \in \mathbb{N}$,

$$\binom{n}{0} + \binom{n}{1}2 + \binom{n}{2}2^2 + \binom{n}{3}2^3 + \cdots + \binom{n}{n-1}2^{n-1} + \binom{n}{n}2^n = 3^n.$$

Solution: The Binomial Theorem is in the textbook. Apply it to $a = 1$ *and* $b = 3$.