Loyola University Chicago
Math 201, Section 001, Fall 2009

Quiz 6 Sample 2

Name (print): ______________________ **Signature:** ______________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. Find the decimal expansion of $\frac{49}{132}$. Clearly show the repeating digits.

$$
\begin{array}{cccccccc}
 & & 0 & . & 3 & 7 & 1 & 2 \\
 & 4 & 9 & : & 1 & 3 & 2 & \\
 & - & 0 & & & & & \\
 & 4 & 9 & 0 & & & & \\
- & 3 & 9 & 6 & & & & \\
 & & 9 & 4 & 0 & & & \\
 & - & 9 & 2 & 4 & & & \\
 & & & 1 & 6 & 0 & & \\
 & & - & 1 & 3 & 2 & & \\
 & & & & 2 & 8 & 0 & \\
 & & & - & 2 & 6 & 4 & \\
 & & & & & 1 & 6 &
\end{array}
$$

The remainder of 16 is a repeat: it showed up two divisions ago. So the 12 digits will now repeat infinitely many times. The answer is $0.3712121212\cdots = 0.37\dot{1}\dot{2}$.

Problem 2. Prove that:

(a) If a and b are rational then $5a - 7b$ is rational.

(b) If x is irrational then $\sqrt[3]{x+1}$ is irrational.

Solution:

(a) Since a and b are rational, there exist $x, y, u, v \in \mathbb{Z}$, $y, v \neq 0$ such that $a = x/y$, $b = u/v$. Then

$$5a - 7b = 5\frac{x}{y} - 7uv = \frac{5xv - 7uy}{yv}$$

and $5xv - 7uy, yv \in \mathbb{Z}$, $yv \neq 0$. Hence $5a - 7b$ is rational.

(b) Prove the contrapositive: If $\sqrt[3]{x+1}$ is rational then x is rational. We have, for some $a, b \in \mathbb{Z}$, $b \neq 0$,

$$\sqrt[3]{x+1} = \frac{a}{b}, \quad x + 1 = \frac{a^3}{b^3}, \quad x = \frac{a^3 - b^3}{b^3}.$$

Since $a^3 - b^3 \in \mathbb{Z}$ and $b^3 \in \mathbb{Z}$. $b^3 \neq 0$, x is rational. The contrapositive is shown. Since the contrapositive is equivalent to the original implication, we are done.

Problem 3. Write down an explicit formula for a bijection from natural numbers divisible by 17 to $\mathbb{Z}$.

Solution: First, a bijection from all natural numbers to $\mathbb{Z}$*:*

$$g(x) = \begin{cases} (x-1)/2 & \textit{if} \quad x \textit{ is odd} \\ -x/2 & \textit{if} \quad x \textit{ is even} \end{cases}$$

This maps 1 *to* 0*,* 2 *to* -1*,* 3 *to* 1*,* 4 *to* -2*,* 5 *to* 2*,* 6 *to* -3*, etc. Now notice that if you take all natural numbers divisible by* 17 *and you divide each of them by* 17*, you'll get all natural numbers; in other words,* $h(x) = x/17$ *is a bijection from natural numbers divisible by* 17 *to natural numbers. Now compose* h *with* g*, and get a bijection from natural numbers divisible by* 17 *to* $\mathbb{Z}$*:*

$$f(x) = \begin{cases} (x/17-1)/2 & \textit{if} \quad x/17 \textit{ is odd} \\ -(x/17)/2 & \textit{if} \quad x/17 \textit{ is even} \end{cases}$$

Problem 4. Suppose that x is a rational number such that x^2 is an integer. Prove that x is an integer.

Solution: Since x *is rational, there exist* $a, b \in \mathbb{Z}$*,* $b \neq 0$*,* $gcd(a, b) = 1$ *such that* $x = \frac{a}{b}$*. Then* $x^2 = \frac{a^2}{b^2}$*.* x^2 *is an integer, so* $x^2 = c$ *for some* $c \in \mathbb{Z}$*. Thus* $\frac{a^2}{b^2} = c$ *and* $a^2 = b^2c$*. Clearly,* $c|a^2$*. Because* $gcd(a^2, b^2) = 1$*,* $a^2|c$*. (This is right, since Proposition 2.28 in the textbook says so.) Because* $c|a^2$ *and* $a^2|c$*, it must be that* $c = a^2$*. (It can not be that* $c = -a^2$ *because* $c \geq 0$*.) Since* $c = a^2$*, we have* $a^2 = b^2a^2$ *and either* $a = 0$*, in which case* $x = 0$ *and* x *is rational, or* $a \neq 0$ *and thus* $b = \pm 1$*. Then* $x = \pm a$ *and so* x *is an integer.*

There are other ways to do this, for example using prime number decompositions of a *and* b*.*

Note that the proof given above used the fact that $gcd(a, b) = 1$ *implies* $gcd(a^2, b^2) = 1$*. It would be OK to just use it, but if you wanted to justify this, here is one way to do it.* $gcd(a, b) = 1$ *iff there exist integer* x, y *such that* $ax + by = 1$*. Then* $(ax + by)^4 = 1$*, and so*

$$1 = a^4x^4 + 4a^3x^3by + 6a^2x^2b^2y^2 + 3axb^3y^3 + b^4y^4 = a^2(a^2x^4 + 4ax^3by) + b^2(6a^2x^2y^2 + 3axby^3 + b^2y^4)$$

which in turn implies that $gcd(a^2, b^2) = 1$*.*