Loyola University Chicago
Math 201, Section 001, Fall 2009

Quiz 6 Sample

Name (print): ____________________ **Signature:** ____________________

You have 30 minutes. Show your work. Notes not allowed! Problems are on both sides of this sheet.

Problem 1. Is the number $3.2457457457457\ldots$ rational or not? If yes, write it as a fraction. If not, give reasons.

Solution: the number is rational, because its decimal expansion is repeating. Let $x = 3.2457457457457\ldots$. *Then*

$$10x = 32.457457457457\ldots,$$

$$10000x = 32457.457457457\ldots,$$

and consequently,

$$10000x - 10x = 32425.$$

Hence

$$x = \frac{32425}{9990}.$$

Problem 2. Prove that $\sqrt[4]{6}$ is not a rational number.

Solution: proof by contradiction. Suppose that $\sqrt[4]{6}$ *is rational. Then there exist integer* a, b, $b \neq 0$, $gcd(a,b) = 1$ *such that* $\sqrt[4]{6} = \frac{a}{b}$. *Then* $6b^4 = a^4$, *and since* $6 = 2 \cdot 3$, $2|a^4$. *Because 2 is prime,* $2|a$, *that is,* $a = 2c$ *for some integer* c. *Then* $6b^4 = (2c)^4$, *hence* $3b^4 = 8c^4$. *Since* $2|8$, $2|3b^4$. *As before, since 2 is prime,* $2|3$ *(which is false) or* $2|b^4$. *The latter must hold, so, again because 2 is prime,* $2|b$. *Hence* $2|a$ *and* $2|b$ *which contradicts* $gcd(a,b) = 1$. *Hence it was wrong to suppose that* $\sqrt[4]{6}$ *is rational, i.e.,* $\sqrt[4]{6}$ *is irrational.*

Problem 3. Write down an explicit formula for a bijection from $\mathbb{Z}$ to $\mathbb{N}$.

Solution: check lecture notes.

Problem 4. Suppose that the coefficients a, b, c of the quadratic equation $ax^2 + bx + c = 0$ are rational and that the equation has two distinct solutions. How many of these solutions can be rational? Give all necessary examples and proofs.

Solution: It could be that both solutions are rational. For example, $x^2 - 1 = 0$. It could be that neither solution is rational. For example, $x^2 - 2 = 0$. It can not happen that one solution is rational and the other one is not. Indeed, let x_1 be a rational solution and x_2 be an irrational solution. Then

$$ax^2 + bx + c = a(x - x_1)(x - x_2) = ax^2 - a(x_1 + x_2)x + ax_1x_2$$

So $b = -a(x_1 + x_2)$ and (since $a \neq 0$ — why is that?) the right hand expression $-a(x_1 + x_2)$ is not rational (why?). But b was assumed rational. Impossible! Contradiction!