Nonlinear Systems and Elements of Control

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April 23, 2013

1 Brief review of basic differential equations

A *differential equation* is an equation involving a function and one or more of the function's derivatives. An *initial value problem* consists of a differential equation and *initial conditions* specifying the value/values of the function and/or of the function's derivatives at a certain point.

Example 1.1 The height h, as a function of time t, of a body falling due to force of gravity and without friction is described by the differential equation

$$\frac{d^2h}{dt^2} = g, \tag{1}$$

where g is the constant acceleration (in metric system, $g = -9.81m/s^2$). In different notation, $h''(t) = g$. Integration yields

$$h'(t) = gt + c, \quad h(t) = \frac{1}{2}gt^2 + ct + d,$$

where c and d are real constants. Hence, $h(t) = \frac{1}{2}gt^2 + ct + d$ is a *solution* to (1). The heigh h as above, of a body which at time $t = 0$ is at height 40 and has velocity 5, is described by the initial value problem

$$\frac{d^2h}{dt^2} = g, \qquad h(0) = 40,\ \frac{dh}{dt}(0) = 5. \tag{2}$$

Since the general solution to the differential equation in (2) is given by $h(t) = \frac{1}{2}gt^2 + ct + d$, one just needs to determine the constants c and d that cause the solution to satisfy the initial conditions $h(0) = 40$ and $h'(0) = 5$:

$$h(0) = \frac{1}{2}g0^2 + c0 + d = d = 40, \quad h'(0) = g0 + c = 5,$$

and so $d = 40$, $c = 5$, and the unique solution to (2) is $h(t) = \frac{1}{2}gt^2 + 5t + 40$. In more generality, the unique solution to the initial value problem

$$h''(t) = g, \qquad h(0) = h_0,\ h'(0) = v_0 \tag{3}$$

is

$$h(t) = \frac{1}{2}gt^2 + v_0t + h_0. \tag{4}$$

If the values of h and h' are predescribed at time t_0 which is not necessarily 0, one can rely on the general solution $h(t) = \frac{1}{2}gt^2 + ct + d$ and use the initial conditions $h(t_0) = h_0$ and $h'(t_0) = v_0$ to determine c and d:

$$h(t_0) = \frac{1}{2}gt_0^2 + ct_0 + d = h_0,\ h'(t_0) = gt_0 + c = v_0$$

which imply that $c = v_0 - gt_0$, $d = h_0 - \frac{1}{2}gt_0^2 - (v_0 - gt_0)t_0$ and so

$$h(t) = \frac{1}{2}gt^2 + (v_0 - gt_0)t + h_0 - \frac{1}{2}gt_0^2 - (v_0 - gt_0)t_0 = \frac{1}{2}g(t-t_0)^2 + v_0(t-t_0) + h_0,$$

where the last expression is obtained from the previous one through some algebra. Note that the last expression can be deduced from (4) by considering $t - t_0$, the amount of time after t_0. △

Example 1.2 Consider the differential equation

$$y'' + 3y' - 10y = 0. \tag{5}$$

To confirm that $y_1(t) = e^{2t}$ is a solution to (5), take the derivatives $y_1'(t) = 2e^{2t}$, $y_1''(t) = 4e^{2t}$ and plug them into the equation to get $4e^{2t} + 3 \cdot 2e^{2t} - 10e^{2t} = 0$, which is true because $4 + 6 - 10 = 0$. Similarly, one confirms that $y_2(t) = e^{-5t}$ is a solution to (5).

The differential equation (5) is a *linear* differential equation, because it has the form

$$A_n(t)y^{(n)}(t) + A_{n-1}(t)y^{(n-1)}(t) + \cdots + A_1(t)y'(t) + A_0(t)y(t) = Q(t), \tag{6}$$

where $y^{(n)}$ denotes the n-th derivative of y, and A_n, A_{n-1}, $\ldots A_1$, A_0, Q are some functions of t. In (5), $A_2 = 1$, $A_1 = 3$, $A_0 = -10$, while Q and all other A_n are 0. Because $A_n = 0$ for $n > 2$ and $A_2 \neq 0$, (5) is a *second order* equation. Because A_2, A_1, A_0 are constant, (5) is a second order equation with *constant coefficients*. Finally, because the right-hand side is 0, it is a *homogeneous* second order equation with constant coefficients.

The practical aspect of linearity and homogeneity of (5) is that any linear combination of solutions to (5) is a solution to (5). Two previously guessed solutions are $y_1(t) = e^{2t}$, $y_2(t) = e^{-5t}$, but the reasoning below works for any other two solutions y_1, y_2 to For some fixed $\alpha, \beta \in \mathbb{R}$, let

$$y(t) = \alpha y_1(t) + \beta y_2(t).$$

Then

$$\begin{aligned} y'' + 3y' - 10y &= (\alpha y_1 + \beta y_2)'' + 3(\alpha y_1 + \beta y_2)' - 10(\alpha y_1 + \beta y_2) \\ &= \alpha y_1'' + \beta y_2'' + 3(\alpha y_1' + \beta y_2') - 10(\alpha y_1 + \beta y_2) \\ &= \alpha\left(y'' + 3y' - 10y\right) + \beta\left(y'' + 3y' - 10y\right) \\ &= \alpha \cdot 0 + \beta \cdot 0 = 0. \end{aligned}$$

Hence, y is a solution to (5).

With the two solutions $y_1(t) = e^{2t}$, $y_2(t) = e^{-5t}$ to (5), let's solve the initial value problem

$$y'' + 3y' - 10y = 0, \qquad y(0) = 4,\ y'(0) = 1. \tag{7}$$

Neither $y_1(t) = e^{2t}$ nor $y_2(t) = e^{-5t}$ satisfy the initial conditions $y(0) = 4$, $y'(0) = 1$. However, the general solution $y(t) = \alpha y_1(t) + \beta y_2(t)$ might, with an appropriate choice of constants α and β. The condition $y(0) = 4$ turns to $\alpha + \beta =$, the condition $y'(0) = 1$ turns to $2\alpha - 5\beta = 1$. Solving the system of two equations for α and β yields $\alpha = 3$, $\beta = 1$. Hence, $y(t) = 3y_1(t) + y_2(t)$ is the solution to (7).

Finally, note that the solutions $y_1(t) = e^{2t}$ nor $y_2(t) = e^{-5t}$ to (5) can be found by first solving the quadratic *characteristic equation* of to (5): $r^2 + 3r - 10 = 0$, which gives $r = 2$ or $r = -5$. Other homogeneous second order linear differential equations, for which the characteristic equations have a repeated real root or complex roots, are solved similarly. See [1, Chapter 3] or [3, Chapter 4] for details. △

Example 1.3 Consider the differential equation

$$\frac{dx}{dt} = x^2. \tag{8}$$

It is a *separable* equation: all occurrences of x can be moved to one side of the equation while all occurrences of t can be moved to the other side. One obtains

$$\frac{dx}{x^2} = dt, \qquad \int \frac{dx}{x^2} = \int dt, \qquad -\frac{1}{x} = t + c$$

and ultimately,

$$x(t) = \frac{1}{c - t},$$

where c is a constant. Consider the function $y(t) = 7x(t) = \frac{7}{c-t}$. Note that

$$\frac{dy}{dt} = \frac{7}{(c-t)^2} \neq \frac{49}{(c-t)^2} = y^2$$

and thus y is not a solution to (8). Consequently, (8) is not a linear differential equation, which could have been concluded by noticing the x^2 term anyway. △

Example 1.4 Given sufficient room and feed, the population of bunnies, the size of which at time t is $b(t)$, grows exponentially according to

$$b'(t) = \alpha b(t), \tag{9}$$

where $\alpha > 0$ is some constant. The equation (9) is a homogeneous first order linear differential equation, and the solution to it is

$$b(t) = b(0)e^{\alpha t}.$$

Note that the function above is well-defined for all $t \in \mathbb{R}$. Similarly, given sufficient room and no bunnies to eat, the population of wolves, the size of which at time t is $w(t)$, decays exponentially according to

$$w'(t) = -\beta w(t),$$

where $\beta > 0$ is some constant. When the populations of bunnies and wolves interact, bunnies get eaten by wolves, which has a negative effect on the former but a positive effect on the latter. This can be modeled by the system of differential equations

$$b'(t) = \alpha b(t) - \gamma b(t)w(t), \qquad w'(t) = -\beta w(t) + \delta b(t)w(t), \tag{10}$$

where $\gamma, \delta > 0$ are some constants. The terms $\gamma b(t)w(t)$, $\delta b(t)w(t)$ appear because the number of interactions between bunnies and wolves is proportional to the product $b(t)w(t)$. △

Example 1.5 Given insufficient room and feed, the population of bunnies $b(t)$ does not grow exponentially. Rather, it can be modeled by

$$b'(t) = \alpha b(t)\,(m - b(t))\,, \tag{11}$$

where $\alpha, m > 0$ are some constants. Note that when $b(t) > m$ then $b'(t) < 0$, when $0 < b(t) < m$ then $b'(t) > 0$, and when $b(t) < 0$ — which has no practical interpretation in terms of bunnies — $b'(t) < 0$. Furthermore, $b(t) \equiv 0$ and $b(t) \equiv m$ are two constant (equilibrium) solutions to (11). The solution to the initial value problem consisting of (11) and $b(0) = b_0$ decreases from b_0 to m if $b_0 > m$ and increases from b_0 to m if $0 < b_0 < m$. This can be verified explicitly, because (11) is separable and can be solved. As long as $b(t) \neq 0$, $b(t) \neq m$, one obtains

$$\frac{db}{dt} = \alpha b(m-b),\ \ \frac{db}{b(m-b)} = \alpha dt,\ \ \frac{1}{m}\left(\frac{1}{b} + \frac{1}{m-b}\right) db = \alpha dt,\ \ \left(\frac{1}{b} - \frac{1}{b-m}\right) db = \alpha m dt$$

and thus

$$\ln|b| - \ln|b-m| = \alpha m t + c,\ \ \ln\left|\frac{b}{b-m}\right| = \alpha m t + c,\ \ \left|\frac{b}{b-m}\right| = e^c e^{\alpha m t}.$$

Because $\left|\frac{b}{b-m}\right| = \pm\frac{b}{b-m}$, one then obtains, with a constant d which can be positive or negative,

$$\frac{b}{b-m} = d e^{\alpha m t}$$

and solving this for b yields

$$b(t) = \frac{dm}{d - e^{-\alpha m t}}.$$

Consider, for example, an initial condition $b(0) = 3m$. Then $3m = \frac{dm}{d-1}$ yields $d = \frac{3}{2}$ and then $b(t) = \frac{3m}{3-2e^{-\alpha m t}}$, which is a decreasing function with $\lim_{t\to\infty} b(t) = m$. If $b(0) = m/2$ then $d = -1$ and $b(t) = \frac{m}{1+e^{-\alpha m t}}$, which is an increasing function with $\lim_{t\to\infty} b(t) = m$.

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2 Examples of simple control problems

Example 2.1 Consider a cart moving along an infinite track (the x-axis) with no friction. The cart has two rocket engines attached to it; one firing to the right (in the positive direction), one firing to the left (in the negative direction). Let $x(t)$ be the position of the cart along the track at time t, so that $\dot{x}(t)$ is the velocity, and $\ddot{x}(t)$ is the acceleration. For simplicity, suppose that the mass of the cart and the force applied to the cart by either of the rocket engines is 1. The differential equations describing the motion of the cart are as follows:

$\ddot{x}(t) = 0$ if both engines are off;

$\ddot{x}(t) = -1$ if the right engine is on;

$\ddot{x}(t) = 1$ if the left engine is on.

Suppose that, at time 0, $x(0) = 12$ and $\dot{x}(0) = -3$; in other words, that the cart is at $x = 12$ and is moving to the left with speed 3. How should one switch the engine (or engines) on and off in order to make the rocket cart stop at the origin at some time $T > 0$ (i.e., $x(T) = 0$, $\dot{x}(T) = 0$) ? One idea is to do nothing for a while, let the cart roll to the left, then, at some time T_0, fire the left engine and let it run until the velocity is 0. T_0 must be picked carefully, so that the cart stops at 0, not elsewhere. The velocity of the cart after T_0, with the left engine running, solves the initial value problem

$$\dot{x}(T_0) = -3, \quad \ddot{x}(t) = 1$$

and thus, for $t \geq T_0$ and as long as the engine is running,

$$\dot{x}(t) = -3 + t - T_0.$$

We want the cart to stop at some (unknown for now) time T, so we want $\dot{x}(T) = 0$, and thus

$$0 = -3 + T - T_0.$$

Consequently, $T = T_0 + 3$, in common words, the cart stops after 3 seconds of the left engine running. The question now is, what should this T_0 be for the cart to stop at 0, i.e, with $x(T) = 0$. For $t \in [0, T_0]$, the position of the cart solves the initial value problem

$$x(0) = 12, \; \dot{x}(0) = -3, \quad \ddot{x}(t) = 0$$

and thus, for $t \in [0, T_0]$,

$$x(t) = 12 - 3t,$$

and in particular, $x(T_0) = 12 - 3T_0$ and $\dot{x}(T_0) = -3$. For $t \in [T_0, T]$, the position of the cart solves the initial value problem

$$x(T_0) = 12 - 3T_0, \; \dot{x}(T_0) = -3, \quad \ddot{x}(t) = 1$$

and thus, for $t \in [T_0, T]$,

$$x(t) = 12 - 3T_0 - 3(t - T_0) + \frac{1}{2}(t - T_0)^2,$$

and in particular, $x(T) = x(T_0 + 3) = 12 - 3T_0 - 3^2 + \frac{1}{2}3^2$. We need $x(T) = 0$, so $0 = 12 - 3T_0 - 3^2 + \frac{1}{2}3^2$ and consequently, $T_0 = 2.5$. Hence, a winning strategy is:

- $\star$ Do nothing for 2.5 seconds. Then, start the left engine and run it for 3 seconds. Then switch the engine off and keep both engines off forever.

Some questions arise about the strategy above, they are formulated in the problem below.

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Problem 2.2 *The questions below pertain to Example 2.1. First, two questions regarding the strategy* $\star$.

(a) *What is the result of the strategy if the initial condition $x(0) = 12$ was not accurate? For example, what if $x(0) = 11.8$ and the same strategy is used?*

(b) *What is the result of the strategy if the initial condition $x'(0) = -3$ was not accurate? For example, what if $x'(0) = -3.1$ and the same strategy is used?*

More interesting questions arise for other initial conditions.

(c) *Suppose that* $x(0) = 10$, $\dot{x}(0) = -2$. *What is a strategy that makes the rocket cart stop at the origin at some time* $T > 0$.

(d) *Suppose that* $x(0) = 12$, $\dot{x}(0) = -8$. *What is a strategy that makes the rocket cart stop at the origin at some time* $T > 0$.

(e) *If one obtained a particular strategy in (b), can one find another strategy that obtains the same goal with less fuel consumption (less time of fuel burning)?*

(f) *Suppose that* $x(0) = 12$, $\dot{x}(0) = -8$. *What is the infimum of the fuel burning time over all strategies that stop the car at the origin at some time* $T > 0$?

Problem 2.3 *This problem is about driving the rocket cart to the origin in the shortest possible time. (Note: this is very different from using as little fuel as possible, i.e., from firing the engines for as little time as possible.) For that purpose, common sense suggests that coasting, i.e., having both engines off, is never an appropriate thing to do unless the cart is already at the origin and stopped. Hence, the rocket cart moves along the* x*-axis according to the differential equations*

$\ddot{x}(t) = -1$ *if the right engine is on;*

$\ddot{x}(t) = 1$ *if the left engine is on.*

Answer the following questions:

(a) *Suppose that* $x(0) = 10$, $\dot{x}(0) = -2$. *What is a strategy that makes the rocket cart stop at the origin in the shortest possible time?*

(b) *Suppose that* $x(0) = 12$, $\dot{x}(0) = -8$. *What is a strategy that makes the rocket cart stop at the origin in the shortest possible time?*

Example 2.4 Let $z(t)$ denote the temperature in a room with a heater which can be on or off. Let z_{off} be the natural temperature of the room with the heater off and z_{on} the natural temperature of the room with the heater on. If the heater is off, the temperature $z(t)$ evolves according to

$$z'(t) = -\alpha(z(t) - z_{off}),$$

where $\alpha > 0$ is some constant. If the heater is on, the temperature $z(t)$ evolves according to

$$z'(t) = -\beta(z(t) - z_{on}).$$

For simplicity, suppose that $z_{off} = 40$, $z_{on} = 80$, $\alpha = \ln 2$, and $\beta = \ln 1.5$ and furthermore, say that $z(0) = 50$. How should one switch the thermostat on and off in order to raise the temperature to be in between 60 and 65 and later keep the temperature in that range? $\triangle$

3 Standard Form

The standard form of an autonomous differential equation is

$$\dot{x} = f(x), \tag{12}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a function. Similarly, the standard form of an non-autonomous differential equation is

$$\dot{x} = f(x, t), \tag{13}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is a function. The standard form of an autonomous control system is

$$\dot{x} = f(x, u), \tag{14}$$

where $x \in \mathbb{R}^n$ is the state of the system, $u \in \mathbb{R}^k$ is the control input, and $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ is a function.

Example 3.1 To write the differential equation $y'' + 3y' - 10y = 0$ of Example 1.2 in the form (12), let $x_1 = y$, $x_2 = y'$ and note that $\dot{x}_1 = y' = x_2$, which comes from the choice of x_1, x_2 while $\dot{x}_2 = y'' = 10y - 3y' = 10x_1 - 3x_2$, which comes from the differential equation. Then, with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, one obtains

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ 10x_1 - 3x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 10 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Thus $\dot{x} = Ax$ with $A = \begin{pmatrix} 0 & 1 \\ 10 & -3 \end{pmatrix}$ △

Example 3.2 The system of differential equations (10) from the predator prey Example 1.4 becomes

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - \gamma x_1 x_2 \\ -\beta x_2(t) + \delta x_1 x_2 \end{pmatrix} \tag{15}$$

after the change of variables $x_1 = b$, $x_2 = w$. Here, $\dot{x} = f(x)$ with $f(x) = \begin{pmatrix} \alpha x_1 - \gamma x_1 x_2 \\ -\beta x_2(t) + \delta x_1 x_2 \end{pmatrix}$.
△

Example 3.3 The rocket cart in Example 2.1, with x_1 denoting the position and $x_2 = \dot{x}_1$ denoting the velocity of the cart and the control u being either -1, 0, or 1, takes the form

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_2 \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{16}$$

△

4 Linear Systems

This section considers autonomous and homogeneous linear differential equations of first order, or linear systems in short, of the form

$$\dot{x} = Ax \tag{17}$$

where $x \in \mathbb{R}^n$ and A is a real $n \times n$ matrix, i.e., $A \in \mathbb{R}^{n\times n}$.

Recall first that linear differential equations as in (6), when homogeneous, and when brought to standard form, take the shape of (17). Solving linear equations (6) of second order, for example $y''+3y'-10y = 0$ from Example 1.2 can be done without matrix methods. This is discussed in [1, Chapter 3] or [3, Chapter 4].

Note that if $\lambda \in \mathbb{R}$ is an *eigenvalue* of A with an associated *eigenvector* $v \in \mathbb{R}^n$, that is, when $Av = \lambda v$, then $e^{\lambda t}v$ solves (17). Indeed,

$$\dot{(e^{\lambda t}v)} = e^{\lambda t}\lambda v = e^{\lambda t}Av = A\left(e^{\lambda t}v\right).$$

Furthermore, given several such eigenvalues and eigenvectors λ_i, v^i and constants c_i, $i = 1, 2, \ldots, m$, one can easily check that $\sum_{i=1}^{m} c_i e^{\lambda_i t} v^i$ is a solution to (17). Hence, solving liner systems (17) with matrix analysis tools works particularly well when A has n independent eigenvectors, which is in particular true when A has n distinct real eigenvalues. This is illustrated in the example below.

Example 4.1 Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$. To find eigenvalues, set the determinant of $\begin{pmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{pmatrix}$ to 0 and solve for λ. $(1-\lambda)^2 - 4 = 0$ yields two eigenvalues $\lambda_1 = 3$, $\lambda_2 = -1$. To find an eigenvalue v^1, solve $Av^1 = \lambda_1 v^1$, so $(A-3I)v^1 = 0$, so

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Get $v^1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and similarly, get $v^2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$. Note that v^1 and v^2 are linearly independent (this always happens for distinct real eigenvalues — why?) and so any initial value problem

$$\dot{x} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x, \qquad x(0) = x^0$$

can be solved by picking constants c_1, c_2 in the general solution

$$x(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

to match the initial condition $x(0) = x^0$. △

Exercise 4.2 *Find the solution to*

$$\dot{x} = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 1 & 4 \\ 3 & 1 & -4 \end{pmatrix} x, \qquad x(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The situation is less simple when there is no n linearly independent eigenvectors for A. The following fact helps with the analysis of (17).

Fact 4.3 *For every $A \in \mathbb{R}^{n\times n}$ there exists a nonsingular matrix $M \in \mathbb{R}^{n\times n}$ and a matrix $J \in \mathbb{R}^{n\times n}$ in real Jordan form so that*

$$A = MJM^{-1}, \qquad J = M^{-1}AM.$$

Consider the change of coordinates

$$z = M^{-1}x, \qquad x = Mz.$$

Then $\dot{x} = Ax$ turns to $M^{-1}\dot{x} = M^{-1}AMM^{-1}x$, so $(\dot{M^{-1}x}) = J(M^{-1}x)$, and hence

$$\dot{z} = Jz. \tag{18}$$

Advantages of dealing with (18) rather than (17) are due to the form of J.

4.1 Linear systems in 2 dimensions

For 2×2 matrices, the real Jordan form can take the following forms:

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \tag{19}$$

and these forms occur, respectively, when A has two distinct real eigenvalues λ_1, λ_2; when A has a single (repeated) real eigenvalue λ and two distinct eigenvectors; when A has a single (repeated) real eigenvalue λ and one eigenvector; and when A has complex eigenvalues $\alpha + i\beta$, $\alpha - i\beta$. The examples below follow the general discussion in [2].

The case of A having two distinct real eigenvalues λ_1, λ_2 is illustrated first.

Example 4.4 Let

$$A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

so in this case $M = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, and $M^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$. Note that $AM = MJ$ and so

$$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 & 2\cdot 2 \\ 1\cdot 1 & 2\cdot 3 \end{pmatrix}$$

which shows that

$$\begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = 2\cdot\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Thus $\lambda_1 = 1$ is an eigenvalue of A with eigenvector $x^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\lambda_2 = 3$ is an eigenvalue of A with eigenvector $x^2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Note that $M^{-1}x^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $M^{-1}x^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Now, let $\phi(t)$ be a solution to (17). Let $\psi(t) = M^{-1}\phi(t)$ and note that $\psi(t)$ solves (18). Write $\psi(t)$ as $\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix}$. Then (18) turns to

$$\begin{pmatrix}\dot\psi_1(t)\\ \dot\psi_2(t)\end{pmatrix} = \dot{\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix}} = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}\psi_1(t)\\ \psi_2(t)\end{pmatrix} = \begin{pmatrix}\psi_1(t)\\ 2\psi_2(t)\end{pmatrix}$$

and $\psi_1(t)$ satisfies $\dot\psi_1(t) = \psi_1$ while $\psi_2(t)$ satisfies $\dot\psi_2(t) = 2\psi_2$. Consequently, $\psi_1(t) = \psi_1(0)e^t$, $\psi_2(t) = \psi_2(0)e^{2t}$, and so

$$\psi(t) = \begin{pmatrix}\psi_1(0)e^t\\ \psi_2(0)e^{2t}\end{pmatrix} = \psi_1(0)e^t\begin{pmatrix}1\\ 0\end{pmatrix} + \psi_2(0)e^{2t}\begin{pmatrix}0\\ 1\end{pmatrix}$$

Such solutions $\psi(t)$ can then be easily sketched in the z coordinate system, and later translated to the x coordinate system. Explicitly, the solution $\phi(t)$ to (17) is

$$\phi(t) = M\psi(t) = \begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}\begin{pmatrix}\psi_1(0)e^t\\ \psi_2(0)e^{2t}\end{pmatrix} = \begin{pmatrix}\psi_1(0)e^t + 2\psi_2(0)e^{2t}\\ \psi_1(0)e^t + 3\psi_2(0)e^{2t}\end{pmatrix} = \psi_1(0)e^t\begin{pmatrix}1\\ 1\end{pmatrix} + \psi_2(0)e^{2t}\begin{pmatrix}2\\ 3\end{pmatrix}$$

Given the general form

$$c_1e^t\begin{pmatrix}1\\ 1\end{pmatrix} + c_2e^{2t}\begin{pmatrix}2\\ 3\end{pmatrix}$$

of a solution to $\dot x = \begin{pmatrix}-1 & 2\\ -3 & 4\end{pmatrix}x$, the solution to this differential equation with the initial condition $x(0) = \begin{pmatrix}2\\ -1\end{pmatrix}$ is found by picking the constants c_1 and c_2 to satisfy the initial condition. One obtains $\psi_1(0) = 8$, $\psi_2(0) = -3$, and consequently the solution is $8e^t\begin{pmatrix}1\\ 1\end{pmatrix} - 3e^{2t}\begin{pmatrix}2\\ 3\end{pmatrix}$ △

Similar analysis can be done when A has two distinct nonzero real eigenvalues of different signs, for example

$$A = \begin{pmatrix}3 & 2\\ 6 & -1\end{pmatrix} = \begin{pmatrix}-1 & 1\\ 3 & 1\end{pmatrix}\begin{pmatrix}-3 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}-1/4 & 1/4\\ 3/4 & 1/4\end{pmatrix}$$

and when A has two distinct negative eigenvalues, for example

$$A = \begin{pmatrix}-8 & 1\\ -5 & -2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 5\end{pmatrix}\begin{pmatrix}-7 & 0\\ 0 & -3\end{pmatrix}\begin{pmatrix}5/4 & -1/4\\ -1/4 & 1/4\end{pmatrix}$$

The situation when A has a single (repeated) nonzero real eigenvalue λ with two eigenvectors is also similar but not all that interesting: in such a case, $A = \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$. The situation is more interesting when to such a single eigenvalue there corresponds only one eigenvector. The following example illustrates this.

Example 4.5 Let

$$A = \begin{pmatrix}-17 & 9\\ -25 & 13\end{pmatrix} = \begin{pmatrix}3 & 1\\ 5 & 2\end{pmatrix}\begin{pmatrix}-2 & 1\\ 0 & -2\end{pmatrix}\begin{pmatrix}2 & -1\\ -5 & 3\end{pmatrix}$$

Here $\lambda = -2$ is the unique (repeated) eigenvalue of A with eigenvector $x^1 = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$. Solving

$$\dot{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}} = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

yields $z_2(t) = z_2(0)e^{-2t}$, $z_1(t) = (z_1(0) + z_2(0)t)e^{-2t}$. Correspondingly,

$$x(t) = Mz(t) = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} (z_1(0) + z_2(0)t)e^{-2t} \\ z_2(0)e^{-2t} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix} e^{-2t} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} z_2(0)te^{-2t}$$

and consequently

$$x(t) = x(0)e^{-2t} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} (-5x_1(0) + 3x_2(0))te^{-2t}$$

Both z_1 and z_2 tend to 0 as $t \to \infty$, same for x_1 and x_2. $\triangle$

Now, to practice working in more generality, suppose that

$$A = M \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} M^{-1}$$

which corresponds to A having complex eigenvalues $\alpha \pm i\beta$. Then $z = M^{-1}x$ evolves according to

$$\dot{z} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} z.$$

It is illustrative to look at $r = \sqrt{z_1^2 + z_2^2}$, $\theta = \arctan \frac{z_1}{z_2}$.

$$\dot{r} = \frac{2z_1\dot{z}_1 + 2z_2\dot{z}_2}{2\sqrt{z_1^2 + z_2^2}} = \frac{z_1(\alpha z_1 + \beta z_2) + z_2(-\beta z_1 + \alpha z_2)}{\sqrt{z_1^2 + z_2^2}} = \alpha\sqrt{z_1^2 + z_2^2} = \alpha r$$

$$\dot{\theta} = \frac{1}{1 + \left(\frac{z_1}{z_2}\right)^2} \frac{\dot{z}_1 z_2 - z_1\dot{z}_2}{z_2^2} = \frac{1}{z_2^2 + z_1^2} \left((\alpha z_1 + \beta z_2)z_2 - z_1(-\beta z_1 + \alpha z_2)\right) = \beta$$

That is, $\dot{r} = \alpha r$, so r grows or decays exponentially or stays constant depending on α, and $\dot{\theta} = \beta$, so θ changes grows or decays linearly or stays constant depending on β. The actual solutions (see [3, Section 9.6]) are

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} e^{\alpha t}\cos\beta t z_1(0) - e^{\alpha t}\sin\beta t z_2(0) \\ e^{\alpha t}\sin\beta t z_1(0) + e^{\alpha t}\cos\beta t z_2(0) \end{pmatrix} = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \end{pmatrix}$$

and so $z(t) = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} z(0)$. It is fun to verify that this is a solution using matrix notation.

Exercise 4.6 *Let*

$$A = \begin{pmatrix} 21 & -10 \\ 26 & -11 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} 5 & -2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

Find a solution to $\dot{x} = Ax$ with $x(0) = (4, 6)$ by finding the corresponding $z(0)$, using this initial condition to solve $\dot{x} = Jz$, and then finding the solution $x = Mz$.

Exercise 4.7 *Find a general solution to* $\dot{x} = Ax$ *if*

$$A = \begin{pmatrix} -35/6 & 1/6 & 1 \\ 5/6 & -31/6 & 5 \\ -5/3 & -5/3 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 5 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} -6 & 0 & 0 \\ 0 & -2 & -1 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} -5/6 & 1/6 & 0 \\ -1/2 & -1/2 & 1 \\ 1/6 & 1/6 & 0 \end{pmatrix}$$

It remains to discuss the cases of one or both eigenvalues being 0.

4.2 Matrix exponential

Given a $n \times n$ matrix A, the *matrix exponential* of A, e^A, is defined as

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots.$$

Ignring the issues of convergence of the infinite series involved, one can show directly from the definition of e^A that $e^{\lambda I} = e^{\lambda} I$ for any $A \in \mathbb{R}^{n\times n}$, $\lambda \in \mathbb{R}$; $e^{A+B} = e^A e^B$ for any $A, B \in \mathbb{R}^{n\times n}$; $\frac{d}{dt}e^{At} = Ae^{At}$ $A \in \mathbb{R}^{n\times n}$, $t \in \mathbb{R}$; and $e^A = Me^J M^{-1}$, $e^{At} = Me^{Jt}M^{-1}$ if $A = MJM^{-1}$. For example, if $A = MJM^{-1}$ then $A^2 = MJM^{-1}MJM^{-1} = MJ^2M^{-1}$ and, similarly, $A^i = MJ^iM^{-1}$ for $i = 3, 4, \dots$, and consequently

$$\begin{aligned} e^{At} &= I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \dots \\ &= I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots \\ &= I + MJM^{-1}t + \frac{1}{2!}MJ^2M^{-1}t^2 + \frac{1}{3!}MJ^3M^{-1}t^3 + \dots \\ &= MIM^{-1} + MJtM^{-1} + M\frac{1}{2!}J^2t^2M^{-1} + M\frac{1}{3!}J^3t^3M^{-1} + \dots \\ &= M\left(I + Jt + \frac{1}{2!}J^2t^2 + \frac{1}{3!}J^3t^3 + \dots\right)M^{-1} \\ &= Me^{Jt}M^{-1}. \end{aligned}$$

These properties then imply that the solution (assuming there is only one) to the initial value problem $\dot{x} = Ax$, $x(0) = x^0$ is provided by

$$x(t) = e^{At}x^0.$$

Indeed, then $\frac{d}{dt}x(t)\frac{d}{dt}\left(e^{At}x^0\right) = \frac{d}{dt}\left(e^{At}\right)x^0 = Ae^{At}x^0 = Ax(t)$. It is instructive to verify this based on the matrix A in one of the following forms: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, or $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, with the last case handled by looking at $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ first. The real Jordan form representation of A may help with computing e^{At} because

$$e^{At} = e^{MJM^{-1}t} = Me^{Jt}M^{-1}.$$

The special structure of J makes it easier to find e^{Jt} in comparison to a general matrix A. If $J = \mathrm{diag}\{\lambda_i\}$ then $e^{Jt} = \mathrm{diag}\left\{e^{\lambda_i t}\right\}$. If $\lambda \in \mathbb{R}$ is such that $(J - \lambda I)^p = 0$ for some $p \in \mathbb{N}$, then

$$e^{Jt} = e^{\lambda t I}e^{(J-\lambda I)t} = e^{\lambda t}\left(I + (J - \lambda I)t + \frac{1}{2!}(J - \lambda I)^2t^2 + \dots + \frac{1}{(p-1)!}(J - \lambda I)^{p-1}t^{p-1}\right).$$

The advantage of such an expression is that it is not an infinite series; rather, it has finitely many terms.

Example 4.8 Let $J = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$. Then

$$J - 3I = \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ (J-3I)^2 = \begin{pmatrix} (-5)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \ldots, \ (J-3I)^q = \begin{pmatrix} (-5)^q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for every $q > 2$ and hence

$$e^{Jt} = e^{3t}\left(I + \begin{pmatrix} -5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} t + \sum_{i=2}^{\infty} \begin{pmatrix} (-5)^i t^i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right) = e^{3t} \begin{pmatrix} e^{-5t} & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-2t} & 0 & 0 \\ 0 & e^{3t} & te^{3t} \\ 0 & 0 & e^{3t} \end{pmatrix}$$

△

Exercise 4.9 *Find* e^{Jt} *for* $J = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

4.3 Large matrices

Knowing how to solve differential equations $\dot{z} = Jz$ with a 2×2 matrix J in the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

lets one deal with larger matrices J that may show up as a real Jordan form of a large matrix A. For example, suppose that $A = MJM^{-1}$ with

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & \beta & \alpha \end{pmatrix},$$

Then $\dot{z} = Jz$ breaks down into

$$\dot{z}_1 = \lambda_1 z_1, \quad \dot{z}_2 = \lambda_2 z_2, \quad \dot{\begin{pmatrix} z_3 \\ z_4 \end{pmatrix}} = \begin{pmatrix} \lambda_3 & 1 \\ 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} z_3 \\ z_4 \end{pmatrix}, \quad \dot{\begin{pmatrix} z_5 \\ z_6 \end{pmatrix}} = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} z_5 \\ z_6 \end{pmatrix}$$

Solutions to these four separate differential equations are obtained separately, following Section 4.1. With given initial conditions, one obtains

$$z_1(t) = e^{\lambda_1 t} z_1(0), \quad z_2(t) = e^{\lambda_2 t} z_2(0),$$

$$\begin{pmatrix} z_3(t) \\ z_4(t) \end{pmatrix} = e^{\lambda_3 t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z_3(0) \\ z_4(0) \end{pmatrix}, \quad \begin{pmatrix} z_5(t) \\ z_6(t) \end{pmatrix} = e^{\alpha t} \begin{pmatrix} \cos\beta t & -\sin\beta t \\ \sin\beta t & \cos\beta t \end{pmatrix} \begin{pmatrix} z_5(0) \\ z_6(0) \end{pmatrix},$$

and consequently,

$$z(t) = \begin{pmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ e^{\lambda_3 t} \left(z_3(0) + t z_4(0) \right) \\ e^{\lambda_3 t} z_4(0) \\ e^{\alpha t} \left(\cos \beta t z_5(0) - \sin \beta t z_6(0) \right) \\ e^{\alpha t} \left(\sin \beta t z_5(0) + \cos \beta t z_6(0) \right) \end{pmatrix}$$

Unfortunately, the discussion above does not cover all the smaller "blocks" that may show up in a real Jordan form of a matrix. One example of another kind of a "block" is in Example 4.9. Another is:

$$\begin{pmatrix} \alpha & -\beta & 1 & 0 \\ \beta & \alpha & 0 & 1 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix} \tag{20}$$

Exercise 4.10 *Find e^{Jt} if J is given by* (20).

4.4 Non-autonomous linear systems

Exercise 4.10 can be nicely handled by relying on the general framework for solving

$$\dot{x} = Ax + b(t) \tag{21}$$

where, as before, $x \in \mathbb{R}^n$, A is a real $n \times n$ matrix, i.e., $A \in \mathbb{R}^{n \times n}$, and $b : [0, \infty) \to \mathbb{R}^n$ is a vector-valued function. Equipped with the notion of the matrix exponential, the solution goes as follows:

$$\dot{x}(t) - Ax(t) = b(t), \quad e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}b(t), \quad \frac{d}{dt}\left(e^{-At}x(t)\right) = e^{-At}b(t)$$

and consequently $e^{-At}x(t) = \displaystyle\int_0^t e^{-As} b(s)\, ds + c$. With an initial condition $x(0)$, one gets $c = x(0)$, and the solution becomes

$$x(t) = e^{At}\left(x(0) + \int_0^t e^{-As} b(s)\, ds \right).$$

4.5 Qualitative properties of solutions to linear systems

Several properties, of solutions to (17), given by $\dot{x} = Ax$, and (21), given by $\dot{x} = Ax + b(t)$, are summarized below.

Given solutions $x_1, x_2, \ldots, x_k$ to (17) on an interval $[0, T]$ and constants $c_1, c_2, \ldots, c_k$, the function x defined by $x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_k x_k(t)$ is a solution to (17). This also holds for (21) if $c_1 + c_2 + \cdots + c_k = 1$.

In particular, if $x : [0, T] \to \mathbb{R}^n$ is a solution to (17) on some interval, then so is cx, for any constant c. This *homogeneity* property means that local behavior and global behavior of solutions to (17) is the same: for example, the saddle point behavior of solutions to (17) with a 2×2 matrix A with one negative and one positive real eigenvalue, is visible no matter how close one zooms in onto the origin or how far one zooms out.

The result below, which will then lead to several properties of solutions to (17) and (21), relies on the fact that for every $n \times n$ matrix A there exists a constant $k \geq 0$ such that for every vector $x \in \mathbb{R}^n$,

$$\|Ax\| \leq k\|x\|.$$

In more technical language, this constant is the induced norm of A, usually denoted by $\|A\|$, and defined by $\|A\| = \max_{v, \|v\| \leq 1} \|Av\|$.

Lemma 4.11 *Let $k \geq 0$ be such that for every vector $x \in \mathbb{R}^n$, $\|Ax\| \leq k\|x\|$. If $y : [0,T] \to \mathbb{R}^n$ is a solution to* (17) *then, for every $t \in [0,T]$,*

$$\|y(t)\| \leq e^{kt}\|y(0)\|.$$

Proof. Consider the function $\alpha : [0,T] \to [0,\infty)$ given by $\alpha(t) = \|y(t)\|^2$. Then α is differentiable, because y and the norm squared are, and

$$\frac{d}{dt}\alpha(t) = 2y(t)\cdot\dot{y}(t) = 2y(t)\cdot Ay(t) \leq 2\|y(t)\|\|Ay(t)\| \leq 2\|y(t)\|k\|y(t)\| = 2k\|y(t)\|^2 = 2k\alpha(t),$$

where the first inequality just says that the dot product of two vectors u and v is bounded above by $\|u\|\|v\|$. Hence $\frac{d}{dt}\alpha(t) \leq 2k\alpha(t)$ and consequently $\alpha(t) \leq e^{2kt}\alpha(0)$. Then, $\|y(t)\| \leq e^{kt}\|y(0)\|$. $\square$

First consequence of Lemma (4.11) is that the size of a solution x to (17) is bounded above, for any t in the domain of x, by an exponential function of t. Thus, there are no solutions $x : [0,T) \to \mathbb{R}^n$ to (17) such that $\lim_{t\to T}\|x(t)\| = \infty$. In other words, solutions to autonomous and homogeneous linear systems do not experience *finite-time blow-up*. This is in contrast to simple nonlinear differential equations, for example $\frac{dx}{dt} = x^2$ from Example 1.3. For $x(0) > 0$, solutions have the form $x(t) = \dfrac{1}{\frac{1}{x(0)} - t}$ and so $\lim_{t\to\frac{1}{x(0)}}\|x(t)\| = \infty$.

Let $x_1, x_2 : [0,T] \to \mathbb{R}^n$ be two solutions to (21) with $x_1(0) = x_2(0)$. Consider $y : [0,T] \to \mathbb{R}^n$ defined by $y(t) = x_1(t) - x_2(t)$, which is a solution to (17). Then, for some $k \geq 0$, $\|y(t)\| \leq e^{kt}\|y(0)\| = 0$, and consequently $x_1(t) = x_2(t)$ for all $t \in [0,T]$. That is, solutions to (21) are unique. This is in contrast to simple nonlinear differential equations, see Example 6.1.

Let $x_1, x_2 : [0,T] \to \mathbb{R}^n$ be two solutions to (21). Consider $y : [0,T] \to \mathbb{R}^n$ defined by $y(t) = x_1(t) - x_2(t)$, which is a solution to (17). Then, for some $k \geq 0$, $\|y(t)\| \leq e^{kt}\|y(0)\|$, and consequently, for every $t \in [0,T]$,

$$\|x_1(t) - x_2(t)\| \leq e^{kt}\|x_1(0) - x_2(0)\| \leq e^{kT}\|x_1(0) - x_2(0)\|,$$

where the constant e^{kT} depends on the matrix A, the length T of the interval $[0,T]$, but not on particular solutions x_1, x_2. One can conclude that, subject to their existence, the solutions to (21) depend uniformly continuously on initial conditions: for every $\varepsilon > 0$, every $T > 0$, there exists $\delta > 0$ (which can be taken to be $e^{-kT}\varepsilon$) such that for every initial point x_0 and every initial point x_0' with $\|x_0' - x_0\| < \delta$, one has $\|x'(t) - x(t)\| < \varepsilon$ for all $t \in [0,T]$, where $x, x' : [0,T] \to \mathbb{R}^n$ are solutions to (21) with initial conditions x_0, x_0'.

Problem 4.12 *Let*

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

be two solutions to the differential equation $\dot{x} = Ax$ *on the interval* $[0, T]$ *with* $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$. *Let* $W(t)$ *be the determinant of*[1]

$$\begin{pmatrix} x_1(t) & y_1(t) \\ x_2(t) & y_2(t) \end{pmatrix}.$$

(a) *Show that*

$$\frac{dW}{dt} = \det \begin{pmatrix} \dot{x}_1(t) & \dot{y}_1(t) \\ x_2(t) & y_2(t) \end{pmatrix} + \det \begin{pmatrix} x_1(t) & x_1(t) \\ \dot{x}_2(t) & \dot{y}_2(t) \end{pmatrix}.$$

(b) *Show that*

$$\frac{dW}{dt} = (a + d)W.$$

(c) *Prove that either* $x(t)$ *and* $y(t)$ *are linearly independent for every* $t \in [0, T]$ *or* $x(t)$ *and* $y(t)$ *are linearly dependent for every* $t \in [0, T]$.

5 Introduction to feedback control

The rocket cart as described in Example 2.1 moves according to $\ddot{x}(t) = 0$ if both engines are off, $\ddot{x}(t) = -1$ if the right engine is on, and $\ddot{x}(t) = 1$ if the left engine is on. In the example, it was established that to make the cart stop at 0 given initial conditions $x(0) = 12$ and $\dot{x}(0) = -3$, one applies the strategy $\star$ which results in the position of the card $x(t)$ solving the differential equation $\ddot{x}(t) = u(t)$, where the function $u : [0, \infty) \to \{-1, 0, 1\}$ is given by

$$u(t) = \begin{cases} 0 & \text{if} \quad t \in [0, 2.5) \\ 1 & \text{if} \quad t \in [2.5, 5.5) \\ 0 & \text{if} \quad t \in [5.5, \infty) \end{cases}$$

The solution $x : [0, \infty) \to \mathbb{R}$ is then $x(t) = 10 - 3t$, $t \in [0, 2.5)$; $x(t) = 4.5 - 3(t - 2.5) + \frac{1}{2}(t - 2.5)^2$, $t \in [2.5, 5.5)$; and $x(t) = 0$, $t \in [5.5, \infty)$. Questions (a) and (b) in Problem 2.2 showed that the desired outcome, the cart resting at 0 for all large enough t, is sensitive to small changes in initial conditions. Using the same $u(t)$ with the initial condition $x(0)$ close but not equal to 12 results in the cart resting but not at 0; using this $u(t)$ with the initial condition $\dot{x}(0)$ close but not equal to -3 results in the cart moving with constant speed for all $t \geq 5.5$. In other words, different initial conditions require different strategies; in fact strategies significantly different as underlined by question (d) in Problem 2.2. Would not it be lovely to have one formula that determines when to fire which engine, independently of the initial conditions? In other words, is there a function $k : \mathbb{R}^2 \to \{-1, 0, 1\}$ so that all solutions to the differential equation $\ddot{x} = k(x, \dot{x})$ get to 0 and stop there? The problem below attempts to find such a formula with an additional goal of minimizing the time it takes the cart to get to 0.

Problem 5.1 *This problem revisits the issue of driving the rocket cart to the origin in the shortest possible time, treated before in Problem 2.3. Do the following:*

(a) *Knowing that* $x(t) = x_0 + v_0 t \pm 0.5t^2$, $v(t) = v_0 \pm t$, *depending on whether the left engine or the right engine is on, sketch two pictures, one for the case of the left engine being on, one for the case of the right engine being on, showing the trajectories of the rocket cart (its position and velocity) in the xv-plane.*

[1] This function is called the *Wronskian* of $x(t)$ and $y(t)$.

(b) *Superimpose the two pictures and sketch the results of strategies from Problem 2.3 (a) and (b) in the superimposed picture.*

(c) *Design a feedback control law for driving the cart to the origin in the shortest possible time. That is, come up with a rule that determines whether the left or the right engine should be running based on the state of the rocket cart, i.e., based on the current position and velocity of the cart.*

The answer to Problem 5.1 (c) is this: if the current state (x, v) of the cart is $(0, 0)$, that is, if the cart is resting at the origin, have both engines of. If not,

- use $a = -1$, that is, fire the right engine, if $v \geq 0$ and $x \geq -\frac{1}{2}v^2$ or if $v <$ and $x > \frac{1}{2}v^2$;
- use $a = 1$, that is, fire the left engine, if $v \leq 0$ and $x \leq \frac{1}{2}v^2$ or if $v > 0$ and $v < -\frac{1}{2}v^2$.

A different way to formulate this is to consider a function $\alpha(x) = \begin{cases} \sqrt{-2x} & \text{if} \quad x < 0 \\ -\sqrt{2x} & \text{if} \quad x \geq 0 \end{cases}$ and say that one should have $a = -1$ if $v > \alpha(x)$; one should have $a = 1$ if $v < \alpha(x)$; and if $v = \alpha(x)$ then have $a = -1$ if $x < 0$ and $a = -1$ if $x > 0$. Consider the function

$$k(x, v) = \begin{cases} -1 & \text{if} \quad v > \alpha(x) \text{ or } v = \alpha(x), x < 0 \\ 1 & \text{if} \quad v < \alpha(x) \text{ or } v = \alpha(x), x > 0 \\ 0 & \text{if} \quad x = 0, v = 0 \end{cases}$$

The task of controlling the rocket cart in a way that makes it stop at the origin in the shortest possible time results in the state of the cart, consisting of the position x and the velocity $v = \dot{x}$, solving the differential equation $\dot{x} = k(x, v)$. Note that the right-hand side of this differential equation is given by a discontinuous function.

5.1 Linear feedback control problems

Problem 5.1 and the discussion surrounding it considered a rocket cart the engines of which can cause acceleration of -1, 0, or 1 and the the goal of stopping the cart at the origin in the shortest possible time. The resulting strategy is a discontinuous function of the carts current position and velocity. Now, let us insist on finding a continuous function of the carts current position and velocity that achieves the goal of stopping the cart at the origin but not necessarily in the shortest possible time. In fact, lets weaken the goal further: we wish to have the carts position and velocity to converge to 0, but strengthen the condition about the strategy: we want it to be a linear function of the current position and velocity. Of course, this is impossible with the "old" rocket engines; so lets suppose that an upgraded engines are available, which can cause the cart to have any acceleration, positive or negative, large or small. We arrive at the following problem:

Example 5.2 Find a function $k(x, v) = ax + bv$ such that setting $u = k(x, v)$ in the differential equation $\ddot{x} = u$ results in a differential equation the solutions of which (and the derivatives of which) converge to 0. When written in the standard form, the differential equation turns to

$$\dot{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} = \begin{pmatrix} x_2 \\ u \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (ax_1 + bx_2) = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

From Section 4.1, it follows that a linear differential equation $\dot{x} = Ax$ has all of its solutions converging to 0 in these cases:

(i) A has two real eigenvalues λ_1, λ_2, not necessarily distinct, and both are negative;

(ii) A has complex eigenvalues $\alpha \pm i\beta$ and $\alpha < 0$.

The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ solve the quadratic equation $\lambda^2 - b\lambda - a = 0$. Case (i) occurs if $b^2 + 4a \geq 0$ and both $\frac{b-\sqrt{b^2+4a}}{2}$ and $\frac{b+\sqrt{b^2+4a}}{2}$ are negative, which occurs when $b + \sqrt{b^2 + 4a}$ is negative. Now, $b + \sqrt{b^2 + 4a} < 0$ if $b^2 + 4a \geq 0$ and $a < 0$. Case (ii) occurs if $b^2 + 4a < 0$, which implies that $a < 0$, and $\frac{b}{2} < 0$. The second inequality comes from seeing that when $b^2 + 4a < 0$, the complex solutions to the quadratic equation are $\dfrac{b \pm i\sqrt{|b^2 + 4a|}}{2}$. Thus, all solutions to the differential equation $\dot{x} = Ax$ converge to 0 if $a < 0$ and $b < 0$.

$\triangle$

6 Introduction to nonlinear differential equations and nonlinear systems

Several properties of linear systems were collected in Section 4.5. For autonomous and homogeneous linear systems $\dot{x} = Ax$ solutions are unique, have at most exponential growth, and depend continuously on initial conditions. Several of these properties can fail for even simple nonlinear differential equations but, as it is proven later, still do hold for many nonlinear differential equations. Of course, there is little hope for a linear combination of solutions to a nonlinear differential equation to be a solution to that differential equation, just as one should not expect a linear combination of two solutions to a quadratic algebraic equation to be a solution to it.

The example below illustrates that an initial value problem with a nonlinear differential equation can have many, in fact infinitely many, solutions.

Example 6.1 Consider the differential equation

$$\dot{x} = 2\sqrt{x}$$

to be solved with nonnegative solutions. One solution can be guessed $x(t) = 0$ for all $t \geq 0$. Separating the variables, under the condition that $x \neq 0$, yields $x(t) = (t + c)^2$. For the initial condition $x(0) = 0$, one obtains a solution $x(t) = t^2$ and this is different than the previously obtained $x(t) \equiv 0$. One can obtain other solutions with $x(0) = 0$ by combining the two solutions found so far. Note that, for any $T \geq 0$, the function

$$x(t) = \begin{cases} 0 & \text{if} \quad t \leq T \\ (t - T)^2 & \text{if} \quad t > T \end{cases}$$

is a solution to $\dot{x} = 2\sqrt{x}$. Consequently, the initial value problem $\dot{x} = 2\sqrt{x}$, $x(0) = 0$ has infinitely many solutions. $\triangle$

Exercise 6.2 *Consider the initial value problem* $\dot{x} = \dfrac{3}{2}x^{\frac{1}{3}}$, $x(0) = 0$.

(a) Find a solution $x(t)$ *on* $[0, \infty)$ *such that* $x(4) = 8$.

(b) Find a solution $x(t)$ *on* $[0, \infty)$ *such that* $x(11) = 8$.

(c) *Find two different solutions* $x_1(t)$ *and* $x_2(t)$ *on* $[0,\infty)$ *such that* $x_1(t) = x_2(t)$ *for all* $t \in [0, 52]$.

(d) *Is there a solution* $x(t)$ *such that* $x(3) = 8$. *Prove your answer.*

Example 1.3 illustrated the case there solutions to a differential equation $\dot{x} = x^2$ *blow-up* in finite time. That is, solutions to this differential equation with $x(0) > 0$ satisfy $\lim_{t\to\frac{1}{x(0)}} \|x(t)\| = \infty$. The example below asks the reader to find a differential equation with a faster blow-up time.

Problem 6.3 *Find a function* $f(x)$ *such that the solution to the initial value problem* $\dot{x} = f(x)$, $x(0) = 1$ *satisfies* $\lim_{t\to 0.1} \|x(t)\| = \infty$.

Another feature of nonlinear differential equations is that they can have many isolated equilibria, that is, points where the dynamics are 0. In contrast, a linear system $\dot{x} = Ax$ has an isolated equilibrium at 0 if A is an invertible matrix, or a subspace consisting of equilibria if A is not invertible.

Example 6.4 Consider the differential equation

$$\dot{x} = -x^2(x-1)(x-2)(x-3)^3$$

which has equilibria at 0, 1, 2, and 3. Inspecting the sign of $f(x) = -x^2(x-1)(x-2)(x-3)^3$ lets one determine the behavior of solutions to the differential equation. For example, because $f(x) > 0$ for $x < 0$, solutions with initial points $x(0) < 0$ increase and converge to 0. Because $f(x) > 0$ for $0 < x < 1$, solutions with initial points $0 < x(0) < 1$ increase and converge to 1. Furthermore, the equilibrium point 1 is locally asymptotically stable because $f(x) < 0$ for $1 < x < 2$ and solutions with $1 < x(0) < 2$ decrease and converge to 1. Similar analysis can be done about the equilibria 2 and 3.

Note that the behavior of solutions with initial points $0 < x(0) < 2$ can be verified through a use of $V(x) = (x-1)^2$ which measures the distance squared of x from 1. If $x(t)$ is a solution to the differential equation, then

$$\frac{d}{dt}V(x) = \frac{d}{dt}(x-1)^2 = 2(x-1)\dot{x} = -2(x-1)x^2(x-1)(x-2)(x-3)^3 = -2x^2(x-1)^2(x-2)(x-3)^3.$$

One can now check that the function on the right end of the equation, $2x^2(x-1)^2(x-2)(x-3)^3$, is negative on $(0,2)$ with the exception of the point $x = 1$, where it is 0. Consequently, any solution in the interval $(0,1)$ or $(1,2)$ decreases its distance from 1 as time goes by. △

A big reason to study linear systems carefully is that they can be used to approximate nonlinear systems. This is done in detail for systems in 2 dimensions in the next section. The idea is illustrated below for a one-dimensional equation from the example above. Consider the equilibrium $\overline{x} = 1$. Then $f(1) = 0$, $f'(1) = -8$ and tangent line approximation says that for x near $\overline{x}$, $f(x) \approx f(\overline{x}) + f'(\overline{x})(x - \overline{x})$, and in this case $f(x) \approx -8(x-1)$. If $x(t)$ is a solution to the differential equation with $x(t)$ near $\overline{x}$ then considering $y(t) = x(t) - \overline{x}$, in this case $y(t) = x(t) - 1$, gives a differential equation for y: $\dot{y} \approx f'(\overline{x})y$, in this case $\dot{y} \approx -8y$. It can be expected that behavior of y resembles the behavior of solutions to $\dot{y} = -8y$. For this differential equation, $y = 0$ is an asymptotically stable equilibrium, and this suggests that for $\dot{x} = -x^2(x-1)(x-2)(x-3)^3$, the equilibrium $\overline{x} = 1$ is asymptotically stable. Similar linear approximation can be done at $\overline{x} = 2$, to discover that $\overline{x} = 2$ is an unstable equilibrium. At $\overline{x} = 0$ and $\overline{x} = 3$, $f'(\overline{x}) = 0$ and no useful information is obtained.

6.1 Nonlinear systems in 2 dimensions

This section suggests how one can try to sketch the behavior of solutions to nonlinear differential equations in two dimensions. This is desired, as frequently, solving such equations is hard or just impossible. Throughout the section, $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a function given as

$$f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}.$$

First, one may want to at least roughly determine the directions in which the solutions flow. This can be done by first determining the *nullclines*, that is, curves in the plane along which $\dot{x}_1 = 0$, so curves given by $f_1(x_1, x_2) = 0$, or curves along which $\dot{x}_2 = 0$, so curves given by $f_2(x_1, x_2) = 0$. Then, one determines the regions in the plane where the solutions move up and right ($\dot{x}_1 > 0$, $\dot{x}_2 > 0$), up and left ($\dot{x}_1 < 0$, $\dot{x}_2 > 0$), down and right ($\dot{x}_1 < 0$, $\dot{x}_2 > 0$), or down and left ($\dot{x}_1 < 0$, $\dot{x}_2 < 0$).

Example 6.5 Consider the differential equation

$$f(x_1, x_2) = \begin{pmatrix} x_2 - x_1^2 \\ -6x_1 + x_2 + x_1^3 \end{pmatrix}.$$

Then $\dot{x}_1 = x_2 - x_1^2$ and so $\dot{x}_1 = 0$ if $x_2 = x_1^2$, $\dot{x}_1 > 0$ if $x_2 > x_1^2$, and $\dot{x}_1 < 0$ if $x_2 < x_1^2$. Furthermore, $\dot{x}_2 = -6x_1 + x_2 + x_1^3$ and so $\dot{x}_2 = 0$ if $x_2 = -x_1^3 + 6x_1$, $\dot{x}_2 > 0$ if $x_2 > -x_1^3 + 6x_1$, and $\dot{x}_2 < 0$ if $x_2 < -x_1^3 + 6x_1$. Consequently, for example, in the region where $x_1 > 0$ and $x_1^2 < x_2 < -x_1^3 + 6x_1$, one has $\dot{x}_1 > 0$, $\dot{x}_2 < 0$, and so the solutions move down and right. △

The points at which the nullclines intersect, that is, points where $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, are equilibria. More precisely, an *equilibrium* of $\dot{x} = f(x)$ is a point $\overline{x}$ such that $f(\overline{x}) = 0$. Each equilibrium point $\overline{x}$ gives rise to a constant solution $x(t) = \overline{x}$ for all $t \in [0, \infty)$ to $\dot{x} = f(x)$. Equilibria can be categorized into isolated equilibria and non-isolated equilibria. *Isolated equilibria* are equilibria $\overline{x}$ such that a sufficiently small neighborhood of $\overline{x}$ does not contain any other equilibria. *Non-isolated equilibria* are equilibria which are not isolated, and so, any neighborhood of a non-isolated equilibrium $\overline{x}$ contains an equilibrium different from $\overline{x}$.

Example 6.6 Consider

$$f(x_1, x_2) = \begin{pmatrix} (x_1 - x_2)(1 - x_1^2 - x_2^2) \\ (x_1 + x_2)(1 - x_1^2 - x_2^2) \end{pmatrix}.$$

Equilibria occur at any $\overline{x}$ with $1 - \overline{x}_1^2 - \overline{x}_2^2 = 0$, and so, at every point $\overline{x}$ of the unit circle. Each such $\overline{x}$ is a non-isolated equilibrium. Another nullcline where $\dot{x}_1 = 0$ is the line $x_1 - x_2 = 0$ and another nullcline where $\dot{x}_2 = 0$ is the line $x_1 + x_2 = 0$. These two nullclines intersect at $(0, 0)$, which is an isolated equilibrium. △

Further information about the behavior of solutions to the differential equation near isolated equilibria can be obtained through linearization. Let the functions f_1 and f_2 defining f be continuously differentiable. Near a point $(\overline{x}_1, \overline{x}_2)$, the function f can be approximated by

$$f(x_1, x_2) \approx \begin{pmatrix} f_1(\overline{x}_1, \overline{x}_2) + \dfrac{\partial f_1}{\partial x_1}(\overline{x}_1, \overline{x}_2)(x_1 - \overline{x}_1) + \dfrac{\partial f_1}{\partial x_2}(\overline{x}_1, \overline{x}_2)(x_2 - \overline{x}_2) \\ f_2(\overline{x}_1, \overline{x}_2) + \dfrac{\partial f_2}{\partial x_1}(\overline{x}_1, \overline{x}_2)(x_1 - \overline{x}_1) + \dfrac{\partial f_2}{\partial x_2}(\overline{x}_1, \overline{x}_2)(x_2 - \overline{x}_2) \end{pmatrix}$$

Consider now the differential equation $\dot{x} = f(x)$ and suppose that $\overline{x} = (\overline{x}_1, \overline{x}_2)$ is an equilibrium, that is, $f(\overline{x}) = 0$. If $x(t) = (x_1(t), x_2(t))$ is a solution to this differential equation then the function

$$y(t) = \begin{pmatrix} x_1(t) - \overline{x}_1 \\ x_2(t) - \overline{x}_2 \end{pmatrix}$$

satisfies $\dot{x} = \dot{y}$ and, if $x(t)$ is near $\overline{x}$, it also satisfies

$$\dot{y}(t) \approx \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\overline{x}_1, \overline{x}_2) y_1 + \frac{\partial f_1}{\partial x_2}(\overline{x}_1, \overline{x}_2) y_2 \\ \frac{\partial f_2}{\partial x_1}(\overline{x}_1, \overline{x}_2) y_1 + \frac{\partial f_2}{\partial x_2}(\overline{x}_1, \overline{x}_2) y_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\overline{x}_1, \overline{x}_2) + \frac{\partial f_1}{\partial x_2}(\overline{x}_1, \overline{x}_2) \\ \frac{\partial f_2}{\partial x_1}(\overline{x}_1, \overline{x}_2) + \frac{\partial f_2}{\partial x_2}(\overline{x}_1, \overline{x}_2) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \nabla f(\overline{x}) y(t)$$

where $\nabla f(\overline{x})$ is the Jacobian of f at $\overline{x}$,

$$\nabla f(\overline{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\overline{x}_1, \overline{x}_2) + \frac{\partial f_1}{\partial x_2}(\overline{x}_1, \overline{x}_2) \\ \frac{\partial f_2}{\partial x_1}(\overline{x}_1, \overline{x}_2) + \frac{\partial f_2}{\partial x_2}(\overline{x}_1, \overline{x}_2) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Thus, one can expect the behavior of $y(t)$ when $y(t)$ is small to resemble the behavior of solutions to the linear system

Consequently, one can expect the behavior of solutions $x(t)$ to $\dot{x} = f(x)$ when $x(t)$ is near an equilibrium $\overline{x}$ to resemble the behavior of solutions $y(t)$ to $\dot{y} = \nabla f(\overline{x}) y$. This lets one classify the equilibria $\overline{x}$ of

$$\dot{x} = f(x) \tag{22}$$

based on the properties of

$$\dot{y} = \nabla f(\overline{x}) y, \tag{23}$$

and so, based on the eigenvalues of $\nabla f(\overline{x})$ at the equilibria. Let $\overline{x}$ be such an equilibrium, i.e., let $f(\overline{x}) = 0$.

- $\overline{x}$ is a stable node for (22) if the origin is a stable node for (23), i.e., if $\nabla f(\overline{x})$ has two distinct real negative eigenvalues;
- $\overline{x}$ is an unstable node for (22) if the origin is an unstable node for (23), i.e., if $\nabla f(\overline{x})$ has two distinct real positive eigenvalues;
- $\overline{x}$ is a saddle point for (22) if the origin is a saddle point for (23), i.e., if $\nabla f(\overline{x})$ has two real eigenvalues with opposite signs;
- $\overline{x}$ is a stable focus for (22) if the origin is a stable focus for (23), i.e., if $\nabla f(\overline{x})$ has complex eigenvalues $\alpha \pm i\beta$ with $\alpha < 0$;
- $\overline{x}$ is an unstable focus for (22) if the origin is an unstable focus for (23), i.e., if $\nabla f(\overline{x})$ has complex eigenvalues $\alpha \pm i\beta$ with $\alpha > 0$.

Example 6.7 Consider the differential equation from Example 6.5. To find equilibria, set $f(x) = 0$, get $x_2 - x_1^2 = 0$, $-6x_1 + x_2 + x_1^3 = 0$; first equation gives $x_2 = x_1^2$, using this in the second equation gives $-6x_1 + x_1^2 + x_1^3 = x_1(x_1 - 2)(x_1 + 3) = 0$. Hence the equilibria are $(0, 0)$, $(2, 4)$, and $(-3, 9)$. The Jacobian is

$$\nabla f(x) = \begin{pmatrix} -2x_1 & 1 \\ -6 + 3x_1^2 & 1 \end{pmatrix}.$$

At the equilibrium $\overline{x} = (0,0)$, the Jacobian is $\nabla f(\overline{x}) = \begin{pmatrix} 0 & 1 \\ -6 & 1 \end{pmatrix}$, the characteristic polynomial is $-\lambda(1-\lambda) + 6 = \lambda^2 - \lambda + 6$, and the eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{23}}{2}$. Consequently, this equilibrium is an unstable focus. At $\overline{x} = (2,4)$, $\nabla f(\overline{x}) = \begin{pmatrix} -4 & 1 \\ 6 & 1 \end{pmatrix}$, the characteristic polynomial is $\lambda^2 + 3\lambda - 6$, the eigenvalues are $\frac{1}{2}\left(-3 \pm \sqrt{33}\right)$, and so this equilibrium is a saddle point. Similarly, $\overline{x} = (-3,9)$ turns out to be a saddle point. △

Exercise 6.8 *Consider*

$$f(x_1, x_2) = \begin{pmatrix} x_2 - x_1^3 \\ x_2^2 - x_1 \end{pmatrix}.$$

Find nullclines, equilibria, classify the equilibria, and sketch the behavior of solutions as well as you can.

Example 6.9 Consider the differential equation from Example 6.6. The only isolated equilibrium occurs at $\overline{x} = (0,0)$. The linearization at $\overline{x} = (0,0)$ can be found quickly by just ignoring all terms in f that are not linear — this does not work at equilibria which are not $(0,0)$. One has

$$f(x_1, x_2) = \begin{pmatrix} x_1 - x_1^3 - x_1x_2^2 - x_2 + x_1^2x_2 - x_2^3 \\ x_1 - x_1^3 - x_1x_2^2 + x_2 - x_1^2x_2 - x_2^3 \end{pmatrix}$$

and so the linearized differential equation is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_1 + x_2 \end{pmatrix}.$$

Of course, one can calculate the Jacobian directly and obtain $\nabla f(\overline{x}) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$. Eigenvalues are $\lambda = 1 \pm i$, and so $\overline{x}$ is an unstable focus. △

Further information about the behavior of solutions can be sometimes obtained by considering polar coordinates. If $x(t) = (x_1(t), x_2(t))$ is a solution to $\dot{x} = f(x)$ and $r(t) = \sqrt{x_1(t)^2 + x_2(t)^2}$ and $\theta = \arctan \dfrac{x_2(t)}{x_1(t)}$, then $\dot{r} < 0$ indicates solutions moving towards the origin, $\dot{r} > 0$ indicates solutions moving away from the origin, while $\dot{\theta} < 0$ indicates clockwise rotation, $\dot{\theta} > 0$ indicates counterclockwise rotation. Trying to get information through polar coordinates is natural when some circular motion is expected or even when a circle plays a special role for f.

Example 6.10 Consider the differential equation from Example 6.6. Every point on the unit circle is an equilibrium. Consider then solutions in polar coordinates. One obtains

$$\begin{aligned} \frac{d}{dt}\left(r^2\right) &= \frac{d}{dt}\left(x_1^2 + x_2^2\right) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(x_1 - x_2)(1-r^2) + 2x_2(x_1+x_2)(1-r^2) \\ &= 2r^2(1-r^2) \end{aligned}$$

and because $\dfrac{d}{dt}\left(r^2\right) = 2r\dot{r}$, one gets $\dot{r} = r(1-r)(1+r)$. Hence $\dot{r} > 0$ if $0 < r < 1$ and $\dot{r} < 0$ if $r > 1$. It is already known, from Example 6.6, that every point with $r = 1$ is an equilibrium, so not surprisingly $\dot{r} = 0$ if $r = 1$. The same comment applies to the unique point with $r = 0$. Similarly one can calculate

$$\dot{\theta} = \frac{d}{dt}\left(\arctan\frac{x_2}{x_1}\right) = \frac{1}{1+\frac{x_2^2}{x_1^2}}\frac{\dot{x}_2 x_1 - x_2\dot{x}_1}{x_1^2} = 1 - r^2.$$

Consequently, inside the unit circle solutions move counterclockwise and outside the unit circle the solutions move clockwise. △

7 Existence, uniqueness of solutions, etc.

7.1 Existence and uniqueness of solutions — Lipschitz f case

Given a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$, suppose that a function $x : [0, T] \to \mathbb{R}^n$ satisfies the following integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\, ds. \tag{24}$$

Then, the function x is a solution to the initial value problem

$$\dot{x} = f(x), \quad x(0) = x_0. \tag{25}$$

Indeed, $x(0) = x_0 + \int_0^0 f(x(s))\, ds = x_0$, and the fundamental theorem of calculus suggests that

$$\dot{x}(t) = \frac{d}{dt}\left(x_0 + \int_0^t f(x(s))\, ds\right) = \frac{d}{dt}\int_0^t f(x(s))\, ds = f(x(t)).$$

On the other hand, any differentiable function $x : [0, T] \to \mathbb{R}^n$ is an integral of its derivative:

$$x(t) - x(0) = \int_0^t \dot{x}(s)\, ds,$$

and so if x is a solution to the initial value problem (25), which ensures that $x(0) = x_0$ and $\dot{x}(s) = f(x(s))$, then x satisfies the integral equation (24).

Note that given any function $x : [0, T] \to \mathbb{R}^n$, the integral formula (24) leads to a new function, say $x'(t)$, defined by $x'(t) = x_0 + \int_0^t f(x(s))$. As noted above, if it turns out that $x'(t) = x(t)$ for all $t \in [0, T]$, then x is a solution to (25). Examples below illustrate what happens if one applies this integral formula repeatedly to functions that are not solutions to (25).

Example 7.1 Consider the initial value problem

$$\dot{x} = cx, \quad x(0) = x_0$$

in $\mathbb{R}$ and the following iterative procedure: let $x_1(t) = x_0$ for all $t \in [0, \infty)$ and, for $n = 1, 2, \ldots$, given $x_n : [0, \infty) \to \mathbb{R}$ define $x_{n+1} : [0, \infty) \to \mathbb{R}$ by

$$x_{n+1}(t) = x_0 + \int_0^t c x_n(s)\, ds.$$

Then

$$x_1(t) = x_0 + \int_0^t cx_0\, ds = x_0(1+ct), \quad x_2(t) = x_0 + \int_0^t cx_0(1+cs)\, ds = x_0\left(1+ct+\frac{c^2t^2}{2}\right)$$

and, in general,

$$x_n(t) = x_0\left(1+ct+\frac{(ct)^2}{2}+\cdots+\frac{(ct)^n}{n!}\right).$$

Note that as $n \to \infty$, $x_n(t) \to x_0e^{ct}$ and x_0e^{ct} happens to be the solution to the initial value problem. △

Example 7.2 Consider the initial value problem

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

in $\mathbb{R}^2$ and the following iterative procedure: let $x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for all $t \in [0,\infty)$ and, for $n = 1, 2, \ldots$, given $x_n : [0,\infty) \to \mathbb{R}$ define $x_{n+1} : [0,\infty) \to \mathbb{R}$ by

$$x_{n+1}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x_n(s)\, ds.$$

Then

$$x_1(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1 \\ -t \end{pmatrix}$$

$$x_2(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -s \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ -1 \end{pmatrix} ds = \begin{pmatrix} 1-t^2/2 \\ -t \end{pmatrix}$$

$$x_3(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1-s^2/2 \\ -s \end{pmatrix} ds = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_0^t \begin{pmatrix} -s \\ -1+s^2/2 \end{pmatrix} ds = \begin{pmatrix} 1-t^2/2 \\ -t+t^3/6 \end{pmatrix}$$

In the limit, one obtains

$$\begin{pmatrix} 1-t^2/2+t^4/4!-t^6/6!+\ldots \\ -t+t^3/3!-t^5/5!+\ldots \end{pmatrix} = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

△

Exercise 7.3 *Repeat the iterative procedure of Example 7.1 starting with $x_1(t) = x_0 + t$.*

Exercise 7.4 *Repeat the iterative procedure of Example 7.1 for the initial value problem*

$$\dot{x} = x^2, \quad x(0) = 1.$$

Pick your own $x_1(t)$.

In the examples and exercises above — examples and exercises which involve differential equations that one can solve by hand — the iterative procedure

$$x_{n+1}(t) = x_0 + \int_0^t f(x_n(s))\, ds \tag{26}$$

yields a sequence of functions which converge to a solution of the initial value problem (25). This suggests that maybe the iterative procedure can be used to show the existence of solutions to a general initial value problem. Roughly, if the iterations produce a sequence of functions which converge (in an appropriate sense) and the limit (in an appropriate sense) of this sequence satisfies (24) then the limit is a solution to (25). To make this precise and the proof (somewhat) rigorous, some background material is needed.

7.1.1 Lipschitz continuity and contractions

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is *Lipschitz continuous* if there exists a constant $L > 0$ such that for every $x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\| \leq L\|x - y\|. \tag{27}$$

Any constant L for which (27) holds is a *Lipschitz constant* for f. The basic example of a Lipschitz function is a linear function $f(x) = Ax$ for some matrix $A \in \mathbb{R}^{n\times n}$. Because for every matrix A there exists a constant $L > 0$ such that $\|Av\| \leq L\|v\|$ for all $v \in \mathbb{R}^n$, for every $x, y \in \mathbb{R}^n$,

$$\|f(x) - f(y)\| = \|Ax - Ay\| = \|A(x - y)\| \leq L\|x - y\|.$$

Another basic example comes from functions $f : \mathbb{R} \to \mathbb{R}$ with bounded derivative. Suppose that, for some $L > 0$, $|f'(x)| \leq L$ for all $x \in \mathbb{R}$. Mean Value Theorem says that for all $x, y \in \mathbb{R}$, $x \neq y$, there exists z in between x and y such that

$$\frac{f(x) - f(y)}{x - y} = f'(z).$$

Then

$$|f(x) - f(y)| = |f'(z)(x - y)| = |f'(z)||x - y| \leq L|x - y|.$$

Similarly, a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with constant L if the Jacobian matrix ∇f is bounded in the sense that, for every $x, v \in \mathbb{R}^n$, $\|\nabla f(x)v\| \leq L\|v\|$.

Example 7.5 The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \dfrac{x}{1 + x^2}$ is Lipschitz continuous. Indeed, it is differentiable with $f'(x) = \dfrac{1 - x^2}{(1 + x^2)^2}$ and the function $|f'(x)| = \dfrac{|1 - x^2|}{(1 + x^2)^2}$ is continuous and approaches 0 when $|x| \to \infty$, hence, $|f'(x)|$ is bounded. In fact, because $|1 - x^2| \leq |1| + |-x^2| = 1 + x^2$, $|f'(x)| \leq \dfrac{1}{1 + x^2} \leq 1$, so f is Lipschitz continuous with constant 1.

$\triangle$

Exercise 7.6 *Show that the following functions $f : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous: $f(x) = |x|$, $f(x) = \arctan x$.*

Exercise 7.7 *Show that if $f, g : \mathbb{R}^n \to \mathbb{R}^n$ are Lipschitz continuous then $f + g$ is Lipschitz continuous. Find an example where f and g are Lipschitz continuous with constant $L > 0$ and $f + g$ is Lipschitz continuous with a constant smaller than L.*

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ which is Lipschitz continuous with constant $L < 1$ is called a *contraction*. The name illustrates the fact that the distance between $f(x)$ and $f(y)$ is less than the distance between x and y.

Theorem 7.8 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction. Then*

(a) There exists exactly one fixed point for f, that is, a point $\overline{x}$ such that $f(\overline{x}) = \overline{x}$.

(b) For every initial point $x_0 \in \mathbb{R}^n$, the sequence of points x_n defined by $x_{n+1} = f(x_n)$ is convergent and such that $\lim_{n\to\infty} x_n = \overline{x}$.

Proof. Let $L \in (0,1)$ be a Lipschitz constant for f. Suppose that there are two fixed points for f, $\overline{x} \neq \overline{y}$. Then

$$\|\overline{x} - \overline{y}\| = \|f(\overline{x}) - f(\overline{y})\| \leq L\|\overline{x} - \overline{y}\|,$$

where the equality holds because $\overline{x}$, $\overline{y}$ are fixed points and the inequality comes from the definition of a contraction. Because $\|\overline{x}-\overline{y}\| \neq 0$, one obtains $1 \leq L$ which contradicts $L < 1$. Consequently, there is at most one fixed point for f. Showing that it exists entails proving (b). Take any $x_0 \in \mathbb{R}^n$ and consider the sequence as defined in (b). Then, for $n = 1, 2, \ldots$,

$$\|x_{n+1} - x_n\| = \|f(x_n) - f(x_{n-1}\| \leq L\|x_n - x_{n-1}\| \leq L^n\|x_1 - x_0\|,$$

where the last inequality comes from repeating the argument n times. Furthermore, for any $m > n$,

$$\begin{array}{rcl}
\|x_m - x_n\| & = & \|x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots + x_{n+1} - x_n\| \\
& \leq & \|x_m - x_{m-1}\| + \|x_{m-1} - x_{m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
& \leq & L^{m-1}\|x_1 - x_0\| + L^{m-2}\|x_1 - x_0\| + \cdots + L^n\|x_1 - x_0\| \\
& = & \left(L^{m-1} + L^{m-2} + \cdots + L^n\right)\|x_1 - x_0\| \\
& = & L^n\left(L^{m-n-1} + L^{m-n-2} + \cdots + 1\right)\|x_1 - x_0\| \\
& \leq & L^n \dfrac{1}{1-L}\|x_1 - x_0\|
\end{array}$$

Consequently, x_n is a Cauchy sequence, meaning that for every $\varepsilon > 0$ there exists $N > 0$ such that for all $m > n > N$, $\|x_m - x_n\| < \varepsilon$. In a complete space, and $\mathbb{R}^n$ is a complete space, Cauchy sequences have limits. Thus the limit of x_n exists, let's call it $\overline{x}$. Then

$$\overline{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f\left(\lim_{n\to\infty} x_{n-1}\right) = f(\overline{x})$$

and so $\overline{x}$ is a fixed point for f. The proof is finished. $\square$

7.1.2 Contractions on spaces of functions

This section discusses Lipschitz functions and contractions not on $\mathbb{R}^n$ but rather on the space of continuous functions on an interval. To avoid confusion, the term *mapping* (rather than function) will be used for the association, to each continuous function f on an interval, of another continuous function Gf on that interval.

Example 7.9 For every function $x : [0,1] \to \mathbb{R}$ let $Gx : [0,1] \to \mathbb{R}$ be the function defined by

$$Gx(t) = x(1-t).$$

So, if $x(t) = 7$ for all $t \in [0,1]$ then $Gx(t) = t$ for all $t \in [0,1]$; if $x(t) = t^2 - 1$ then $Gf(t) = (1-t)^2 - 1 = t^2 - 2t$; etc. Note that if f is continuous then so is Gf, and that for every function f, $G^2 f = G(Gf) = f$. $\triangle$

A mapping G assigning to each continuous function $x : [0,T] \to \mathbb{R}^n$ another continuous function $Gx : [0,T] \to \mathbb{R}^n$ is *Lipschitz continuous* with *Lipschitz constant* $L > 0$ if, for every pair of continuous functions $x_1, x_2 : [0,T] \to \mathbb{R}^n$,

$$\max_{t\in[0,T]} \|Gx_1(t) - Gx_2(t)\| \leq L \max_{t\in[0,T]} \|x_1(t) - x_2(t)\|$$

and it is a *contraction* if it is Lipschitz continuous with constant $L < 1$.

Example 7.10 The mapping G from Example (7.9) is Lipschitz continuous with constant 1. In fact, because for every continuous function $x : [0,1] \to \mathbb{R}$,

$$\max_{t\in[0,T]} |x(t)| = L \max_{t\in[0,T]} |x(1-t)|,$$

one has

$$\max_{t\in[0,T]} \|Gx_1(t) - Gx_2(t)\| = \max_{t\in[0,T]} \|x_1(t) - x_2(t)\|.$$

$\triangle$

Theorem 7.11 *Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz continuous with constant $L > 0$ and let $x_0 \in \mathbb{R}^n$. Define a mapping G from the space of continuous functions $x : [0,T] \to \mathbb{R}^n$ to the space of continuous functions $x : [0,T] \to \mathbb{R}$ by*

$$Gx(t) = x_0 + \int_0^t f(x(s))\, ds.$$

Then G is Lipschitz continuous with constant LT. If $LT < 1$, then G is a contraction.

Proof. Consider any two continuous functions $x_1, x_2 : [0,T] \to \mathbb{R}^n$. Then

$$\begin{aligned}
\|Gx_1(t) - Gx_2(t)\| &= \left\| x_0 + \int_0^t f(x_1(s))\, ds - x_0 - \int_0^t f(x_2(s))\, ds \right\| \\
&= \left\| \int_0^t f(x_1(s)) - f(x_2(s))\, ds \right\| \\
&\leq \int_0^t \|f(x_1(s)) - f(x_2(s))\|\, ds \leq \int_0^t L\|x_1(s) - x_2(s)\|\, ds \\
&\leq \int_0^t L \max_{s\in[0,T]} \|x_1(s) - x_2(s)\|\, ds = \max_{s\in[0,T]} \|x_1(s) - x_2(s)\| \int_0^T L\, ds \\
&= LT \max_{t\in[0,T]} \|x_1(t) - x_2(t)\|
\end{aligned}$$

and consequently

$$\max_{t\in[0,T]} \|Gx_1(t) - Gx_2(t)\| \leq LT \max_{t\in[0,T]} \|x_1(t) - x_2(t)\|.$$

$\square$

Exercise 7.12 *Verify directly that the sequence of functions $x_n(t)$ from Example 7.1, when considered on an interval $[0,T]$, satisfies the inequality*

$$\max_{t\in[0,T]} \|x_{n+1}(t) - x_n(t)\| \leq cT \max_{t\in[0,T]} \|x_n(t) - x_{n+1}(t)\|$$

for every $n = 1, 2, \ldots$.

Problem 7.13 *The mapping G from the space of real-valued continuous function on $[0,2]$ to the space of real-valued continuous functions on $[0,2]$ is given by*

$$Gx(t) = 3x(t) - \int_0^t \sin(x(s))\, ds.$$

Show that G is Lipschitz continuous and find its Lipschitz constant.

Problem 7.14 *Verify that the mapping G on the space of continuous functions from $[0, T]$ to $\mathbb{R}$, defined by*

$$Gx(t) = \int_0^t \sqrt{x(s)}\, ds$$

is not Lipschitz continuous, no matter how small $T > 0$ is.

7.1.3 Proof of existence and uniqueness

Fact 7.15 *If G is a contraction on the space of continuous functions $x : [0, T] \to \mathbb{R}^n$ then there exists exactly one fixed point for G, that is, a function $\overline{x} : [0, T] \to \mathbb{R}^n$ such that $Gx(t) = x(t)$ for every $t \in [0, T]$.*

Theorem 7.16 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function. Then, for every $x_0 \in \mathbb{R}^n$ there exists a solution $\overline{x} : [0, \infty) \to \mathbb{R}^n$ to the initial value problem* (25) *and this solution is unique on every interval $[0, T]$.*

Proof. Let $L > 0$ be a Lipschitz constant for f and let $T > 0$ be such that $LT < 1$. Consider the mapping G defined in Theorem 7.11. According to that theorem, G is a contraction on the space of continuous functions $x : [0, T] \to \mathbb{R}^n$. Fact 7.15 implies that G has a unique fixed point, let's denote it by $\overline{x}_1$. The discussion at the beginning of Section 7.1 implies that $\overline{x}_1 : [0, T] \to \mathbb{R}^n$ is then the unique solution to (25) on $[0, T]$. To obtain the solution $\overline{x}$ on $[0, \infty)$, proceed recursively. Given $\overline{x}_n : [0, T] \to \mathbb{R}^n$, repeated the argument above for the initial value problem $\dot{x} = f(x)$, $x(0) = \overline{x}_n(T)$ to obtain a unique solution $\overline{x}_{n+1} : [0, T] \to \mathbb{R}^n$. Then, the desired solution on $[0, \infty)$ is obtained by considering $\overline{x}(t) = \overline{x}_n(t - (n-1)T)$ for $t \in [(n-1)T, nT]$. $\square$

The uniqueness claimed in Theorem 7.16 can be dealt with in another way. Suppose that, for some $\tau > 0$, there are two solutions $x_1, x_2 : [0, \tau] \to \mathbb{R}^n$ to the initial value problem (25). Consider the differentiable function $v(t) = \|x_1(t) - x_2(t)\|^2$ and note that $v(0) = 0$ and

$$\begin{aligned}
\frac{d}{dt} v(t) &= 2(x_1(t) - x_2(t)) \cdot (\dot{x}_1(t) - \dot{x}_2(t)) \leq 2\|x_1(t) - x_2(t)\| \|\dot{x}_1(t) - \dot{x}_2(t)\| \\
&= 2\|x_1(t) - x_2(t)\| \|f(x_1(t)) - f(x_2(t))\| \leq 2\|x_1(t) - x_2(t)\| L \|x_1(t) - x_2(t)\| \\
&\leq 2Lv(t)
\end{aligned}$$

Consequently, $v(t) \leq v(0)e^{2Lt} = 0$ and so $\|x_1(t) - x_2(t)\| = 0$.

7.2 Related results and more general cases

The argument carried out after Theorem 7.16 in order to show another way of verifying uniqueness of solutions to the initial value problem (25) generalizes to the following: if f is Lipschitz continuous with constant L, then, for every two solutions x', x'' on $[0, \infty)$ to $\dot{x} = f(x)$, for every $t > 0$,

$$\|x'(t) - x''(t)\| \leq e^{Lt} \|x'(0) - x''(0)\|.$$

As a consequence, one obtains a result about continuous dependence of solutions to $\dot{x} = f(x)$ on initial conditions.

Theorem 7.17 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz continuous function. Then, for every $T > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that any two solutions $x', x'' : [0, T] \to \mathbb{R}^n$ to the differential equation $\dot{x} = f(x)$ such that*

$$\|x'(0) - x''(0)\| < \delta$$

are such that

$$\max_{t \in [0,T]} \|x'(t) - x''(t)\| < \varepsilon.$$

In fact, one can take $\delta = \varepsilon e^{-LT}$, where L is a Lipschitz constant for f.

Note that this result does imply uniqueness of solutions to initial value problems $\dot{x} = f(x)$, $x(0) = x_0$ for every x_0: if, for x', x'' as in the result, $x'(0) = x_0 = x''(0)$ then $x'(t) = x''(t)$ for all $t \in [0, T]$.

The result about continuous dependence of solutions on initial conditions, and the consequence about uniqueness, easily generalizes to the case where the right-hand side of the differential equation depends on t, but is Lipschitz continuous in x uniformly in t. More precisely, if $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is such that there exists $L > 0$ such that for every $t \in \mathbb{R}$, every $x, y \in \mathbb{R}^n$,

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\|,$$

then any two solutions x', x'' on $[0, T]$ to $\dot{x} = f(x)$, for every $t \in [0, T]$, $\|x'(t) - x''(t)\| \leq e^{Lt}\|x'(0) - x''(0)\|$. The proof is essentially identical to that of Theorem 7.17. The reader should verify this.

Theorem 7.16, for a Lipschitz continuous f, ensured the existence and uniqueness of solutions to $\dot{x} = f(x)$ on $[0, \infty)$. In absence of Lipschitz continuity, for example when $f(x) = x^2$, as discussed in Example 1.3, solutions can experience finite-time blow-up, and thus fail to exist on $[0, \infty)$. If f happens to be locally Lipschitz continuous, and $f(x) = x^2$ is such a function, then solutions are unique in an appropriate sense to be specified below. When f is continuous but not Lipschitz continuous, for example if $f(x) = \sqrt{|x|}$, existence can still be ensured, but uniqueness may fail as illustrated in Example 6.1. To be able to state a precise result, a definition is needed. Consider a differential equation

$$\dot{x} = f(x). \tag{28}$$

A solution $x : I \to \mathbb{R}^n$ to (28), where I is an interval beginning at 0, is called *maximal* if there does not exist another solution $y : J \to \mathbb{R}^n$, where J is an interval beginning at 0, such that $I \subset J$, $x(t) = y(t)$ for every $t \in I$, and $I \neq J$. In other words, a solution is maximal if it cannot be extended forward in time.

Theorem 7.18 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Then, for every initial point $x_0 \in \mathbb{R}^n$, there exists a maximal solution x to the initial value problem*

$$\dot{x} = f(x), \quad x(0) = x_0,$$

and either x is defined on $[0, \infty)$ or x is defined on $[0, T)$ for some $T > 0$ and $\lim_{t \to T^-} \|x(t)\| = \infty$. If f is locally Lipschitz continuous, then for every initial point $x_0 \in \mathbb{R}^n$, the maximal solution to the initial value problem is unique.

This result will not be proven here. It is an ambitious exercise to tackle the proof the theorem above for the case of f locally Lipschitz continuous, following what was done for the Lipschitz continuous case in and before Theorem 7.16. The significant difference between the Lipschitz and locally Lipschitz cases is that the functions $\overline{x}_1$, $\overline{x}_2$, $\overline{x}_3$, etc. featured in the proof of Theorem Theorem 7.16 had domains of equal length, so concatenating them results in the interval $[0, \infty)$, while in the locally Lipschitz case, the lengths of domains of $\overline{x}_n$ can decrease as $n \to \infty$, and a concatenation may result in an interval of the form $[0, T)$. For the case of a continuous, but not locally Lipschitz continuous f, the approach relying on contractions is not appropriate and different proofs are needed.

Continuous dependence of solutions on initial conditions cannot be expected for a continuous f when the uniqueness of maximal solutions fails. It can be expected when uniqueness is ensured, for example, when f is locally Lipschitz continuous.

Theorem 7.19 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Suppose that, for an initial point $x_0 \in \mathbb{R}^n$, there exists a unique maximal solution $x : I \to \mathbb{R}^n$ to $\dot{x} = f(x)$, $x(0) = x_0$. Then, for every $T > 0$ such that $[0, T] \subset I$, every $\varepsilon > 0$, there exists $\delta > 0$ such that for every initial point x_0' such that*

$$\|x_0' - x_0\| < \delta,$$

every maximal solution y to $\dot{x} = f(x)$, $x(0) = x_0'$ is defined on $[0, T]$ and

$$\max_{t \in [0,T]} \|y(t) - x(t)\| < \varepsilon.$$

Problem 7.20 *Verify directly the assumptions and the conclusions of Theorem 7.19 for solutions to the differential equation*

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \sqrt{|x_1|} \\ 0 \end{pmatrix},$$

for the initial points $x_0 = (0, 0)$ and $x_0' = (0, \varepsilon)$ with $\varepsilon > 0$.

It was noted already that Theorem 7.16 extends to the case of f depending on t. Similarly, conclusions of Theorem 7.18 hold for the differential equation $\dot{x} = f(t, x)$ if, for every x, the function $f(t, x)$ is piecewise continuous in t and, for every t, the function $f(t, x)$ is continuous or locally Lipschitz continuous in x.

8 Asymptotic stability

Throughout this chapter, a general nonlinear differential equation

$$\dot{x} = f(x) \tag{29}$$

is considered, under the assumption that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. The property of interest is the asymptotic stability of an equilibrium for (29). This property requires that solutions that start near the equilibrium remain near this equilibrium and, furthermore, that they converge to this equilibrium as $t \to \infty$. Such a property is frequently present in physical systems, for example due to dissipation of energy. It is also a property frequently desired in a control system

$$\dot{x} = f(x, u) \tag{30}$$

and possibly can be achieved by appropriate selection of feedback control $u = k(x)$.

8.1 Asymptotic stability

The origin, for the differential equation (29), is

- *Lyapunov stable* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every maximal[2] solution x with $|x(0)| < \delta$ is defined on $[0, \infty)$ and such that $|x(t)| < \varepsilon$ for all $t \in [0, \infty)$;
- *locally attractive* if there exists $\delta > 0$ such that every maximal solution with $|x(0)| < \delta$ is defined on $[0, \infty)$ and such that $\lim_{t\to\infty} x(t) = 0$;
- *locally asymptotically stable* if it is both Lyapunov stable and locally attractive.

If the origin is locally asymptotically stable then the *basin of attraction* of the origin is the set of all initial conditions $x_0 \in \mathbb{R}^n$ such that every maximal solution with $x(0) = x_0$ is defined on $[0, \infty)$ and such that $\lim_{t\to\infty} x(t) = 0$. If the basin of attraction of the locally asymptotically stable origin is equal to $\mathbb{R}^n$, the origin is *globally asymptotically stable*.

For example, the origin is globally asymptotically stable for $\dot{x} = ax$ if and only if $a < 0$. Indeed, every solution has the form $x(t) = x(0)e^{at}$ and so if $a < 0$ then $|x(t)| \leq |x(0)|$ which verifies Lyapunov stability (one can take $\delta = \varepsilon$) and furthermore $\lim_{t\to\infty} x(t) = 0$, no matter what $x(0)$ is, which verifies global asymptotic stability. If $a = 0$, the origin is Lyapunov stable. If $a > 0$, the origin is not Lyapunov stable and not locally attractive: for arbitrarily small $\delta > 0$, the solution with $x(0) = \delta$ is $x(t) = \delta e^{at}$ and $\lim_{t\to\infty} x(t) = \infty$.

For the differential equation $\dot{x} = x(x-1)(x+2)$, the origin is locally asymptotically stable with the basin of attraction $(-2, 1)$. This can be justified without explicitly finding the solutions. The rough reason for local asymptotic stability is that the function $f(x) = x(x-1)(x+2)$ is positive for $x \in (-2, 0)$, so solutions $x(t)$ when in $(-2, 0)$ increase, and $f(x)$ is negative for $x \in (0, 1)$, so solutions $x(t)$ when in $(0, 1)$ decrease.

8.2 Asymptotic stability for linear systems

This section discusses asymptotic stability of the origin for the linear system

$$\dot{x} = Ax \tag{31}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix. Because linear systems can be solved explicitly, asymptotic stability can be verified directly. For example, consider

$$A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}.$$

In Example 4.4 it was shown that the general form of a solution is

$$x(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Unless $c_1 = c_2 = 0$, each such solution satisfies $\|x(t)\| \to \infty$ when $t \to \infty$ and so the origin is not Lyapunov stable and not locally attractive. On the other hand, consider

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix},$$

[2] Maximal solutions are defined above Theorem 7.18.

which has solutions of the form

$$x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Clearly, each such solution $x(t)$ converges to 0 when $t \to \infty$. To verify Lyapunov stability, consider the following calculation (which relies on the fact that $a^2 + 2|a||b| + b^2 \leq 2(a+b)^2$):

$$\begin{array}{rcl} \|x(t)\| & = & \sqrt{(c_1 e^{-t} + 2c_2 e^{-2t})^2 + (c_1 e^{-t} + 3c_2 e^{-2t})^2} \\ & = & \sqrt{\left(c_1^2 e^{-2t} + 2c_1 c_2 e^{-3t} + 4c_2^2 e^{-4t}\right) + \left(c_1^2 e^{-2t} + 2c_1 c_2 e^{-3t} + 9c_2^2 e^{-4t}\right)} \\ & \leq & \sqrt{\left(c_1^2 e^{-2t} + 2|c_1||c_2| e^{-3t} + 4c_2^2 e^{-4t}\right) + \left(c_1^2 e^{-2t} + 2|c_1||c_2| e^{-3t} + 9c_2^2 e^{-4t}\right)} \\ & \leq & \sqrt{\left(c_1^2 + 2|c_1||c_2| + 4c_2^2\right) e^{-2t} + \left(c_1^2 + 2|c_1||c_2| + 9c_2^2\right) e^{-2t}} \\ & \leq & e^{-t} \sqrt{2(c_1 + 2c_2)^2 + (c_1 + 3c_2)^2} \\ & = & e^{-t} \sqrt{2} \|x(0)\|. \end{array}$$

In particular, $\|x(t)\| \leq \sqrt{2}\|x(0)\|$. This shows that the origin is Lyapunov stable: for $\varepsilon > 0$, in the definition of Lyapunov stability, one can take $\delta = \varepsilon/\sqrt{2}$. The computation above suggests that, despite knowing the solutions explicitly, verifying Lyapunov stability may not be very simple. The situation gets more complicated for

$$A = \begin{pmatrix} -17 & 9 \\ -25 & 13 \end{pmatrix}.$$

In Example 4.5 it was shown that the solutions have the form

$$x(t) = x(0)e^{-2t} + \begin{pmatrix} 3 \\ 5 \end{pmatrix} (-5x_1(0) + 3x_2(0)) t e^{-2t}.$$

Clearly, such $x(t)$ converges to 0 when $t \to \infty$. Hence, the origin is globally attractive. Verifying Lyapunov stability — which does hold — directly takes some more effort; the reader should try. Later in this section, and also in Section 8.4, methods to verify asymptotic stability of linear systems without solving them first are discussed. Below, the methods come down to checking the eigenvalues of the matrix A.

Because for linear systems local asymptotic stability is equivalent to global asymptotic stability, as the reader is asked to verify, the adjectives are dropped below and asymptotic stability of (31) is discussed. The key to the exercise below is homogeneity: if $x : [0, \infty) \to \mathbb{R}^n$ is a solution to (31) then so is the function $ax(t)0$, for any constant scalar a.

Exercise 8.1 *Show that if the origin is locally asymptotically stable for* (31) *then the origin is globally asymptotically stable for* (31).

Let $M \in \mathbb{R}^{n \times n}$ be a nonsingular matrix such that $A = MJM^{-1}$ for a matrix J in real Jordan form (recall Fact 4.3 and the discussion of real Jordan forms for 2×2 matrices in Section 4.1). The change of variables $z = M^{-1}x$, equivalently, $x = Mz$ leads to another linear system

$$\dot{z} = Jz \tag{32}$$

with a function $x : [0, \infty) \to \mathbb{R}^n$ solving (31) if and only if the related $z : [0, \infty) \to \mathbb{R}^n$ given by $z(t) = M^{-1}x(t)$ solves (32).

Lemma 8.2 *The origin is asymptotically stable for* (31) *if and only if the origin is asymptotically stable for* (38).

Proof. Suppose that the origin is asymptotically stable for (38), so it is Lyapunov stable and attractive. To verify Lyapunov stability of the origin for (31), pick $\varepsilon > 0$. Let $k_1 > 0$ be such that $\|Mv\| \leq k_1\|v\|$ for every $v \in \mathbb{R}^n$. Let $\varepsilon' = \varepsilon/k_1$ and using Lyapunov stability of the origin for (32) find $\delta' > 0$ such that every solution to (32) with $\|z(0)\| < \delta'$ is such that $\|z(t)\| < \varepsilon'$ for all $t \geq 0$. Now let $k_2 > 0$ be such that $\|M^{-1}v\| \leq k_2\|v\|$ for every $v \in \mathbb{R}^n$ and let $\delta = \delta'/k_2$. Then, every solution to (31) with $\|x(0)\| < \delta$ is such that the solution $z = M^{-1}x$ to (32) satisfies $\|z(0)\| = \|M^{-1}x(0)\| \leq k_2\|x(0)\| < k_2\delta = \delta'$. Then, by the choice of δ', $\|z(t)\| < \varepsilon'$ for all $t \geq 0$. Hence $\|x(t)\| = \|Mz(t)\| \leq k_1\|z(t)\| < k_1\varepsilon' = \varepsilon$ for all $t \geq 0$. This verifies Lyapunov stability of the origin for (31). To verify attractivity of the origin for (31), note that for every $x_0 \in \mathbb{R}^n$, the solution $z : [0,\infty) \to \mathbb{R}^2$ to (32) with $z(0) = M^{-1}x_0$ converges to 0 as $t \to \infty$, and hence the solution $x : [0,\infty) \to \mathbb{R}^2$, which must by given by $x(t) = Mz(t)$, converges to 0 as $t \to \infty$. This verifies attractivity of the origin for (31). □

For a linear system $\dot{x} = Ax$ in $\mathbb{R}^2$, the analysis carried out in Section 4.1 suggests that the origin is globally asymptotically stable if A has two real negative eigenvalues or if A has complex eigenvalues $\alpha \pm i\beta$ with $\alpha < 0$. To rigorously justify this, consider first the case of A having two real negative eigenvalues $\lambda_1 \leq \lambda_2 < 0$, possibly equal to one another, and suppose that $J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then, for any solution $z : [0,\infty) \to \mathbb{R}^2$ to (32),

$$\|z(t)\| = \left\| \begin{pmatrix} z_1(0)e^{\lambda_1 t} \\ z_2(0)e^{\lambda_2 t} \end{pmatrix} \right\| = \sqrt{z_1^2(0)e^{2\lambda_1 t} + z_2^2(0)e^{2\lambda_2 t}} \leq e^{\lambda_2 t}\sqrt{z_1^2(0) + z_2^2(0)} = e^{\lambda_2 t}\|z(0)\|$$

and because $\lambda_2 < 0$, the origin is asymptotically stable for (32).

In the case of of A having a repeated real negative eigenvalue $\lambda < 0$ and when $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, one has, for any solution $z : [0,\infty) \to \mathbb{R}^2$ to (32),

$$\begin{aligned}
\|z(t)\| &= \left\| \begin{pmatrix} (z_1(0) + tz_2(0))\, e^{\lambda t} \\ z_2(0)e^{\lambda t} \end{pmatrix} \right\| = e^{\lambda t}\sqrt{z_1^2(0) + 2tz_1(0)z_2(0) + t^2 z_2^2(0) + z_2^2(0)} \\
&\leq e^{\lambda t}\sqrt{(1+t^2)z_1^2(0) + 2(1+t^2)|z_1(0)||z_2(0)| + (1+t^2)z_2^2(0)} \\
&= e^{\lambda t}\sqrt{1+t^2}\sqrt{z_1^2(0) + 2|z_1(0)||z_2(0)| + z_2^2(0)} \leq e^{\lambda t}\sqrt{1+t^2}\sqrt{2z_1^2(0) + 2z_2^2(0)} \\
&= e^{\lambda t}\sqrt{2(1+t^2)}\|z(0)\|
\end{aligned}$$

Consequently, $\|z(t)\| \leq M\|z(0)\|$ where M is the maximum of $e^{\lambda t}\sqrt{2(1+t^2)}$ over $t \in [0,\infty)$, which verifies Lyapunov stability (take $\delta = \varepsilon/M$) and $\|z(t)\| \to 0$ when $t \to \infty$, which verifies attractivity. The missing steps in this argument are a nice calculus exercise.

Exercise 8.3 *Let* $\lambda < 0$. *Find the maximum of* $\phi(t) = e^{\lambda t}\sqrt{2(1+t^2)}$ *over* $t \in [0,\infty)$ *and verify that* $\lim_{t\to\infty} \phi(t) = 0$. *Also, show that for any* $a, b \in \mathbb{R}$, $a^2 + 2|a||b| + b^2 \leq 2a^2 + 2b^2$.

For the case of A having complex eigenvalues $\alpha \pm i\beta$, and so when $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, the polar coordinates argument carried out in Section 4.1 shows that $\|z(t)\| = e^{\alpha t}\|z(0)\|$. When $\alpha < 0$, this ensures that the origin is asymptotically stable for (32).

On the other hand, if A has a real eigenvalue $\lambda \geq 0$, then the solution $x(t) = e^{\lambda t} v$, where v is any eigenvector associated with λ, shows that the origin is not attractive. Similarly, the equality $\|z(t)\| = e^{\alpha t}\|z(0)\|$ in the case of A having complex eigenvalues $\alpha \pm i\beta$ shows that $\alpha \geq 0$ contradicts attractivity. This, combined with the arguments in the above paragraphs and with Lemma 8.2, implies the following result.

Theorem 8.4 *The linear* 2×2 *system* (31) *has the origin asymptotically stable if and only if the eigenvalues of* A *are real and negative or complex with a negative real part.*

This result is valid for $n \times n$ matrices.

Some of the discussion in this section underlined how directly checking for Lyapunov stability of the origin for the linear system (31) may be somewhat technical. It turns out that attractivity, which is simple to verify if one knows the general form of a solution to (31), is sufficient to guarantee Lyapunov stability. This is a special feature of linear systems and fails in the nonlinear case.

Theorem 8.5 *If the origin is attractive for the linear* $n \times n$ *system* (31) *then the origin is asymptotically stable for* (31).

8.3 Lyapunov functions

Consider a general nonlinear differential equation

$$\dot{x} = f(x) \tag{33}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous.

A continuously differentiable function $V : D \to \mathbb{R}$ is called a *Lyapunov function* for (33) if

- $V(0) = 0$, $V(x) > 0$ if $x \neq 0$, and $\lim_{\|x\| \to \infty} V(x) = \infty$;
- for every $x \neq 0$,
$$\nabla V(x) \cdot f(x) < 0. \tag{34}$$

A function satisfying the first condition above and satisfying $\nabla V(x) \cdot f(x) < 0$ only for $x \neq 0$ on a neighbourhood of 0 is called a *local Lyapunov function*.

For example, if $c < 0$ then $V(x) = x^2$ is a Lyapunov function for $\dot{x} = cx$ on $\mathbb{R}$, because $x^2 \geq 0$, $x^2 = 0$ only if $x = 0$, $x^2 \to \infty$ if $x \to \pm\infty$, and $\nabla V(x) = 2x$ and so $\nabla V(x) \cdot f(x)$ turns to $2xcx = 2cx^2$, and this is negative unless $x = 0$. For the differential equation $\dot{x} = f(x) = x(x-1)(x+2)$, $V(x) = x^2$ is a local Lyapunov function because $\nabla V(x) \cdot f(x) = 2xx(x-1)(x+2) = 2x^2(x-1)(x+2) < 0$ for $x \neq 0$ in $(-2, 1)$.

The implication of the Lyapunov inequality (34) is that, for every solution $x : [0, T] \to \mathbb{R}^n$, the function $V(x(t))$ is decreasing (as a function of time). In other words, the Lyapunov function for (33) is decreasing along nonzero solutions to (33). Indeed, the chain rule yields

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot \dot{x}(t) = \nabla V(x(t)) \cdot f(x(t)),$$

and this is negative unless $x(t) = 0$.

There are two very important facts about Lyapunov functions:

- Checking if a function V is a Lyapunov function for (33) does not require solving (33).
- The existence of a Lyapunov function for (33) ensures asymptotic stability of the origin for (33).

The first fact above is important because solving (33) may be hard or impossible. On the other hand, checking if a given function is a Lyapunov function is frequently simpler. This is illustrated below, in Example 8.6. Of course, there is a related difficulty: finding a Lyapunov function may be hard. The second fact, which is formally stated and proved as is Theorem 8.9, is important because checking for asymptotic stability directly, even if (33) can be solved, need not be easy.

Example 8.6 Consider the differential equation $\dot{x} = f(x)$ with $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$f(x) = \begin{pmatrix} -x_1 + x_1 x_2 \\ -2x_2 - 4x_1^2 \end{pmatrix}.$$

Consider the function

$$V(x) = \frac{1}{2}\left(x_1^2 + x_2^2\right).$$

Then

$$\nabla V(x) \cdot f(x) = x_1(-x_1 + x_1 x_2) + x_2(-2x_2 - 4x_1^2) = -x_1^2 - 2x_2^2 - 3x_1^2 x_2.$$

This quantity is not negative for all $x \neq 0$. For example, consider $x_2 = -1$. Then, the quantity is $2x_1^2 - 2$, which is positive when $|x_1| > 1$. Hence, this V is not a Lyapunov function for $\dot{x} = f(x)$. However, it is a local Lyapunov function: note that if $x_2 > -1/3$ then

$$\nabla V(x) \cdot f(x) = -x_1^2 - 2x_2^2 - 3x_1^2 x_2 < -x_1^2 - 2x_2^2 - x_1^2 = -2x_2^2 \leq 0,$$

and so for all $x \neq 0$ with $x_2 > -1/3$, $\nabla V(x) \cdot f(x) < 0$.

Now, consider another function,

$$V(x) = 2x_1^2 + \frac{1}{2}x_2^2.$$

Then

$$\nabla V(x) \cdot f(x) = 4x_1(-x_1 + x_1 x_2) + x_2(-2x_2 - 4x_1^2) = -4x_1^2 - 2x_2^2.$$

Consequently, $\nabla V(x) \cdot f(x) < 0$ for all $x \neq 0$. Hence V is a Lyapunov function for (33). Theorem 8.9 will show that this implies that the origin is globally asymptotically stable. △

Some preliminary material is required before Theorem 8.9.

$$S_{\leq r} V = \{x \in \mathbb{R}^n \mid V(x) \leq r\}$$

Lemma 8.7 *Let V be a Lyapunov function for (33). Then:*

(a) For every $\varepsilon > 0$ there exists $r > 0$ such that $x \in S_{\leq r}$ implies $|x| < \varepsilon$.

(b) For every $r > 0$ the set $S_{\leq r}$ is closed and bounded.

Exercise 8.8 *Prove Lemma 8.7.*

Theorem 8.9 *If there exists a Lyapunov function for* (33) *then the origin is globally asymptotically stable for* (33)*. If there exists a local Lyapunov function for* (33) *then the origin is locally asymptotically stable for* (33)*.*

Proof. Suppose that V is a Lyapunov function for (33). Lyapunov stability of the origin is shown first. Pick $\varepsilon > 0$. Then, by Lemma 8.7 (a) there exists $r > 0$ such that $x \in S_{\leq r}$ implies $|x| < \varepsilon$. Because V is continuous and $V(0) = 0$, there exists $\delta > 0$ such that $|x| < \delta$ implies $V(x) \leq r$. Consequently, because $V(x(t))$ is decreasing for every solution $x : [0, T] \to \mathbb{R}^n$ of (33), every such solution with $|x(0)| < \delta$ satisfies $V(x(t)) \leq r$ for all $t \in [0, T]$, and hence $|x(t)| < \varepsilon$ for all $t \in [0, T]$. Thus, every maximal solution to (33) with $|x(0)| < \delta$ is bounded by ε, i.e., $|x(t)| < \varepsilon$, and thanks to Theorem 7.18 it must be defined on $[0, \infty)$. This completes the proof of Lyapunov stability.

To prove that every maximal solution to (33) is defined on $[0, \infty)$ and converges to 0 as $t \to \infty$, let x be a maximal solution to (33). Then $V(x(t)) \leq V(x(0))$ for all t in the domain of the solution, and so $x(t) \in S_{\leq r}$ with $r = V(x(0))$. By Lemma 8.7 (b), the set $S_{\leq r}$ is bounded, so the maximal solution x is bounded, so by Theorem 7.18 it is defined on $[0, \infty)$. From boundedness, let $R > 0$ be such that $|x(t)| \leq R$ for all $r \geq 0$.

To see that $\lim_{t\to\infty} x(t) = 0$, suppose that, to the contrary, there exists some $\varepsilon > 0$ and a sequence of times $t_1, t_2, \dots$ with $\lim_{i\to\infty} t_i = \infty$ such that $|x(t_i)| \geq \varepsilon$ for $i = 1, 2, \dots$. Let $\delta > 0$ be related to this $\varepsilon > 0$ as required by Lyapunov stability of the origin. If there was a time τ such that $|x(\tau)| < \delta$, then for all $t \geq \tau$, $|x(t)| < \varepsilon$, which is impossible because $|x(t_i)| \geq \varepsilon$ for $i = 1, 2, \dots$ and $t_i \to \infty$. Hence, the maximal solution $x : [0, \infty) \to \mathbb{R}^n$ satisfies $\delta \leq |x(t)| \leq R$ for all $t \geq 0$. The function $\nabla V(x) \cdot f(x)$ is continuous and negative on the closed and bounded set $S = \{x \,|\, \delta \leq x \leq R\}$ and hence there exists $v > 0$ such that $\nabla V(x) \cdot f(x) \leq -v$ for all $x \in S$. Then

$$\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) \leq -v$$

and consequently

$$V(x(t)) = V(x(0)) + \int_0^t \frac{dV}{dt}(x(s))\, ds < V(x(0)) - vt.$$

This implies that $\lim_{t\to\infty} V(x(t)) = -\infty$, which is impossible because the function V is nonnegative. Hence, the assumption that $\lim_{t\to\infty} x(t) = 0$ fails was wrong. It must be that $\lim_{t\to\infty} x(t) = 0$ and the proof is finished. $\square$

Example 8.10 A closer look at the Lyapunov function $V(x) = 2x_1^2 + \frac{1}{2}x_2^2$ for the differential equation

$$f(x) = \begin{pmatrix} -x_1 + x_1 x_2 \\ -2x_2 - 4x_1^2 \end{pmatrix}$$

from Example 8.6 helps illustrate some elements of the proof of Theorem 8.9. First, the reader should verify that the linearization of f at $x = 0$ yields a linear system $\dot{x}_1 = -x_1$, $\dot{x}_2 = -2x_2$. Hence it should be expected that the origin is locally asymptotically stable for $\dot{x} = f(x)$. Theorem 8.9 confirms this, in fact implies that the origin is globally asymptotically stable. Indeed, Example 8.6 verified that V is a Lyapunov function.

To see better how V helps deal with Lyapunov stability, note that sublevel sets of V, that is, sets $S_{\leq r}$, are ellipses. For example, the set of points x with $V(x) \leq 2$ is the ellipse

$x_1^2 + \dfrac{x_2^2}{4} \le 1$. Note that this ellipse contains the circle, centered at the origin, of radius 1 and is itself contained in the circle, centered at the origin, of radius 2. Because $V(x(t))$ is decreasing along every solution to $\dot{x} = f(x)$, a solution that satisfies $V(x(0)) < 1$ will satisfy $V(x(t)) < 2$ for all $t \ge 0$. Consequently, if a solution satisfies $|x(0)| < 1$ then it also satisfies $|x(t)| < 2$ for all $t \ge 0$. The reader should verify that, similarly, if a solution satisfies $|x(0)| < \varepsilon/2$ then it also satisfies $|x(t)| < \varepsilon$ for all $t \ge 0$, and this verifies Lyapunov stability of the origin, with $\delta = \varepsilon/2$. $\triangle$

8.4 Lyapunov functions for linear systems

Section 8.2 showed that asymptotic stability of a linear system

$$\dot{x} = Ax \tag{35}$$

can be verified directly, even if with some technicalities, because (35) can be solved explicitly. More importantly, the section showed that checking asymptotic stability of a linear system comes down to checking the sign of the eigenvalues, or of the real part of the eigenvalues. Section 8.3 showed that Lyapunov functions can be used to confirm asymptotic stability of a general nolinear differential equation $\dot{x} = f(x)$. This section discusses Lyapunov functions for linear systems. The most important conclusion is that if the origin is asymptotically stable for (35) then there exists a quadratic Lyapunov function confirming this.

First, some examples. The matrix

$$A = \begin{pmatrix} -1 & 2 \\ 0 & -3 \end{pmatrix}$$

has eigenvalues -1 and -3, and so, by Theorem 8.4, the origin is asymptotically stable for (35). The simplest possible guess, $V(x) = \|x\|^2 = x_1^2 + x_2^2$, is a Lyapunov function for (35), which verifies asymptotic stabilty. There are other quadratic Lyapunov functions in this case: if $V(x) = x_1^2 + cx_2^2$ then

$$\nabla V(x) \cdot Ax = 2x_1(-x_1 + 2x_2) + 2cx_2(-3x_2) = -2(x_1 - x_2)^2 - 2(3c - 1)x_2^2$$

and so V is a Lyapunov function if $c > 1/3$.

Exercise 8.11 *Find a Lyapunov function for* (35) *if* $A = \begin{pmatrix} -1 & 10 \\ 0 & -1 \end{pmatrix}$.

As it was verified directly in Section 8.2, the linear system (35) has the origin asymptotically stable when

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}.$$

Finding a Lypaunov function to confirm this is not straightforward. Trying $V(x) = \|x\|^2$ gives $\nabla V(x) \cdot Ax = 2x_1^2 + 2x_1x_2 - 8x_2^2$ and this quantity is positive for many choices of x_1, x_2. One can try $V(x) = ax_1^2 + cx_2^2$, with the constants $a > 0$ and $c > 0$ to be determined, and then

$$\nabla V(x) \cdot Ax = ax_1^2 + 2(-2a + 3c)x_1x_2 - 8cx_2^2.$$

It is impossible to make this quantity negative for all x with the constraint that $a > 0$. Indeed, considering $x = (x_1, 0)$ gives $\nabla V(x) \cdot Ax = ax_1^2 > 0$ for all $x_1 \ne 0$. Note that searching for a Lyapunov function of the form $ax_1^2 + cx_2^2$ is no more general than looking at $x_1^2 + cx_2^2$. Indeed, the former function is a scalar multiple of the latter.

A general form of a quadratic function of $x \in \mathbb{R}^n$ is

$$V(x) = ax_1^2 + 2bx_1x_2 + cx_2^2 \tag{36}$$

for constants $a, b, c \in \mathbb{R}$. The 2 is inserted for convenience; after all, $2b$ is a constant. When could such a function be a candidate for a Lyapunov function? One has $V(0) = 0$; one needs $V(x) > 0$ if $x \neq 0$. Of course, it must be that $a > 0$ and $b > 0$. Further conditions on the constants come from a simple calculus problem. Fix x_1 and consider $\phi(x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$. The minimum of ϕ is $\left(a - \frac{b^2}{c}\right) x_1^2$. For this to be positive if $x_1 \neq 0$, it must be that $a - \frac{b^2}{c} > 0$. Consequently, the function V in the form (36) satisfies $V(x) > 0$ for $x \neq 0$ if and only if $a > 0$, $c > 0$, and $ac - b^2 > 0$. In fact, one can drop the superfluous condition $c > 0$ (as otherwise, thanks to $a > 0$, one would get $ac - b^2 \leq 0$).

A different way to express (36), which will be useful in some linear algebra arguments below, is

$$V(x) = x \cdot Px \quad \text{where} \quad P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}. \tag{37}$$

The matrix P is symmetric: $P^T = P$ and *positive definite*: for every vector $v \neq 0$, $v \cdot Pv > 0$. The paragraph above justified an alternative characterization of a positive definite matrix: a symmetric matrix P, as in (37), is positive definite if an only if $a > 0$ and $ac - b^2 > 0$, i.e., if $a > 0$ and the determinant of P is positive. Note that for V in the form (**??**), $\nabla V(x) = Px$, and so if V is a Lyapunov function for (35) then

$$\nabla V(x) \cdot Ax = (Px) \cdot Ax = x \cdot P^T Ax = x \cdot PAx < 0$$

for all $x \neq 0$.

The existence of quadratic (in the form (36) or, equivalently, (37)) Lypaunov functions for a 2×2 linear system for which the origin is asymptotically stable can be verified through considering the real Jordan form of A. Given A, let M be a nonsingular matrix such that $A = MJM^{-1}$ for a matrix J in real Jordan form. The change of variables $z = M^{-1}x$, equivalently, $x = Mz$ leads to another linear system

$$\dot{z} = Jz \tag{38}$$

with a function $x : [0, \infty) \to \mathbb{R}^n$ solving (35) if and only if the related $z : [0, \infty) \to \mathbb{R}^n$ given by $z(t) = M^{-1}x(t)$ solves (38). Lemma 8.2 showed that asymptotic stability for (38) is equivalent to asymptotic stability for (35). Suppose now that W is a quadratic Lyapunov function for (38) given by

$$W(z) = z \cdot Qz \tag{39}$$

for a symmetric and positive definite 2×2 matrix W. Then $z \cdot QJz < 0$ for $z \neq 0$. However,

$$\begin{aligned} z \cdot QJz &= \left(M^{-1}x\right) \cdot QJ\left(M^{-1}x\right) = x \cdot M^{-T}QJM^{-1}x = x \cdot M^{-T}QM^{-1}MJM^{-1}x \\ &= x \cdot M^{-T}QM^{-1}Ax, \end{aligned}$$

and if this quantity is negative for all $x \neq 0$ (equivalently, for all $z = M^{-1}x \neq 0$), then the function $V(x) = x \cdot Px$ with

$$P = M^{-T}QM^{-1}$$

is a Lyapunov function for (35). The reader should verify that if Q is symmetric then so is P and if Q is positive definite then so it P. In summary, if $W(z) = z \cdot Qz$ is a Lyapunov function for (38) then $V(x) = x \cdot Px$ with $P = M^{-T}QM^{-1}$ is a Lyapunov function for (35). In other words, if the function $W(z) = z \cdot Qz$ is a Lyapunov function for (38) then $V(x) = W(M^{-1}x)$ is a Lyapunov function for (35).

It now remains to see if, at least in the 2×2 case, for every asymptotically stable linear system $\dot{z} = Jz$ where the matrix J is in real Jordan form, there exists a quadratic Lyapunov function. Consider first

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

where $\lambda_1 \leq \lambda_2 < 0$. Then $W(z) = \|z\|^2$ is a Lyapunov function for $\dot{z} = Jz$ (the reader should check!), and $\|z\|^2$ is quadratic, given by (39) with Q being the identity matrix. When

$$J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

it was verified in Section 4.1, by passing to polar coordinates, that $W(z) = \|z\|^2$ is a Lyapunov function for $\dot{z} = Jz$. Finally, if

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\lambda < 0$, the function $W(z) = z_1^2 + cz_2^2$ is a Lyapunov function for $\dot{z} = Jz$ as long as $c > 1/4\lambda^2$. The reader should verify this. In summary, for the 2×2 case, whenever the origin is asymptotically stable for $\dot{x} = Ax$, there exists a quadratic Lyapunov function $W(z) = z \cdot Qz$ for $\dot{z} = Jz$, where J is the real Jordan form of A. Then, as the previous paragraph showed, $V(x) = x \cdot Px$ is a quadratic Lyapunov function for $\dot{x} = Ax$, where $P = M^{-T}QM^{-1}$. This, combined with Theorem 8.9, amounts to the following result.

Theorem 8.12 *The linear* 2×2 *system* (35) *has the origin asymptotically stable if and only if there exists a quadratic Lyapunov function* $V(x) = x \cdot Px$ *for* (35).

This result, just as Theorem 8.4, is valid for $n \times n$ matrices. The usefulness of this result is in the ability to assert the existence of a Lyapunov function for a linear system (35), without the need to find it explicitly, based only on the analysis of the eigenvalues of the matrix A.

Example 8.13 Consider the linear system (35) with

$$A = \begin{pmatrix} -17 & 9 \\ -25 & 13 \end{pmatrix}.$$

As it was discussed in Section 8.2, based on the general form of solutions to this system — which was found in Example 4.5 — one can easily conclude that all solutions converge to 0 as $t \to \infty$. Then, Theorem 8.5 implies that the origin is asymptotically stable for (35). Alternatively, one can check eigenvalues of A and use Theorem 8.4 to conclude asymptotic stability. Theorem 8.12 then implies that there exists a quadratic Lyapunov function $V(x) = x \cdot Px$ for (35).

To find a quadratic Lyapunov function explicitly, consider the real Jordan form of A: $A = MJM^{-1}$ where

$$M = \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}, \quad J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix}.$$

Then $W(z) = \frac{1}{2}\|z\|^2$ is a Lyapunov function for $\dot{z} = Jz$ and so $V(x) = W(M^{-1}x)$ is a Lyapunov function for $\dot{x} = Az$. More explicitly,

$$\begin{aligned} V(x) &= \frac{1}{2}\|M^{-1}\|^2 = \frac{1}{2}\left\| \begin{pmatrix} 2x_1 - x_2 \\ -5x_1 + 3x_2 \end{pmatrix} \right\|^2 = \frac{1}{2}(2x_1 - x_2)^2 + \frac{1}{2}(-5x_1 + 3x_2)^2 \\ &= 14.5x_1^2 - 17x_1x_2 + 5x_2^2. \end{aligned}$$

In the linear algebra form, $W(z) = \frac{1}{2}z \cdot Qz$ with $Q = I$, the identity matrix. Then $V(x) = \frac{1}{2}x \cdot Px$ where $P = M^{-T}QM^{-1} = M^{-T}M^{-1}$, and so

$$P = M^{-T}M^{-1} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 29 & -17 \\ -17 & 10 \end{pmatrix}.$$

Then, indeed, $V(x) = \dfrac{1}{2}x \cdot \begin{pmatrix} 29 & -17 \\ -17 & 10 \end{pmatrix} x = 14.5x_1^2 - 17x_1x_2 + 5x_2^2.$ $\triangle$

Problem 8.14 *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function such that $\phi(x) > 0$ for all $x \in \mathbb{R}^2$. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by*

$$f(x) = \begin{pmatrix} \phi(x)(x_2 - 2x_1) \\ \phi(x)(4x_1 - 5x_2) \end{pmatrix} = \phi(x) \begin{pmatrix} -2 & 1 \\ 4 & -5 \end{pmatrix} x.$$

Show that the origin is asymptotically stable for $\dot{x} = f(x)$.

The section concludes with a discussion of the rate of decrease of a quadratic Lyapunov function $V(x) = \frac{1}{2}x \cdot Px$ for an asymptotically stable linear system (35). It will be now shown that the right-hand side of the Lyapunov inequality $\nabla V(x) \cdot Ax < 0$ is in fact bounded by a negative multiple of $V(x)$ itself. As usual, the details are worked out in the 2×2 case, but the conclusion is true in general. Suppose that the Lyapunov inequality

$$\nabla V(x) \cdot Ax < 0$$

holds for all $x \neq 0$. Equivalently, $x \cdot PAx < 0$, and because

$$(x \cdot PAx)^T = (x^T PAx)^T = x^T A^T P^T (x^T)^T = x^T A^T Px = x \cdot A^T Px,$$

the Lyapunov inequality is equivalent to

$$x \cdot PAx = \frac{1}{2}\left(x \cdot PAx + x \cdot A^T Px\right) x = \frac{1}{2}x \cdot \left(PA + A^T P\right) x < 0. \tag{40}$$

The reason for considering $PA + A^T P$, rather than PA, is that $PA + A^T P$ is a symmetric matrix. One can now ask whether, for some $r > 0$, the Lyapunov inequality implies

$$\nabla V(x) \cdot Ax < -rV(x). \tag{41}$$

The stronger Lyapunov inequality (41) is equivalent to $\frac{1}{2}x \cdot \left(PA + A^T P\right) x < -rx \cdot Px$, and the also to

$$x \cdot \left(\frac{1}{2}PA + \frac{1}{2}A^T P + rP\right) x < 0$$

This holds, for all $x \neq 0$, if the matrix $-\left(\frac{1}{2}PA + \frac{1}{2}A^T P + rP\right)$ is positive definite. To see if $r > 0$ making this possible exists, let

$$P = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \qquad \frac{1}{2}\left(PA + A^T P\right) = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

The matrix P is positive definite, so $a > 0$, $ac - b^2 > 0$. Because (40) holds for all $x \neq 0$, $-(PA + A^T P)$ is positive definite, and so $-\alpha > 0$, $\alpha\gamma - \beta^2 > 0$. The matrix $-\left(\frac{1}{2}PA + \frac{1}{2}A^T P + rP\right)$, equal to

$$\begin{pmatrix} -\alpha - ra & -\beta - rb \\ -\beta - rb & -\gamma - rc \end{pmatrix},$$

is positive definite if $-\alpha - ra > 0$, which holds for all small $r > 0$ because $-\alpha > 0$, and if

$$(-\alpha - ra)(-\gamma - rc) - (-\beta - rb)^2 = \alpha\gamma - \beta^2 + r(a\gamma + \alpha c - 2\beta b) - r^2(ac - b^2) > 0.$$

Because $\alpha\gamma - \beta^2 > 0$, this holds for small enough r. Note also that because $ac - b^2 > 0$, this fails for large r. In summary, there exists $r > 0$ such that (41) holds for all $x \neq 0$.

Theorem 8.15 *The linear* 2×2 *system* (35) *has the origin asymptotically stable if and only if there exists a function* $V(x) = x \cdot Px$ *with symmetric and positive definite matrix* P *and* $r > 0$ *such that* (41) *holds for all* $x \neq 0$.

8.5 Linearization and asymptotic stability

In Section 6.1, linear approximation of a nonlinear differential equation

$$\dot{x} = f(x) \tag{42}$$

near an equilibrium was used to deduce behavior of solutions to (42) near equilibria. This section rigorously justifies some of the conclusions made in Section 6.1. It will be proved that if the origin is asymptotically stable for the linearization of (42) then the origin is locally asymptotically stable for (42).

Given a continuously differentiable $f : \mathbb{R}^n \to \mathbb{R}^n$ and the differential equation (42), consider the linear system

$$\dot{x} = Ax \qquad \text{where } A = \nabla f(0). \tag{43}$$

Theorem 8.16 *Suppose that the origin is an equilibrium for* (42). *If the origin is asymptotically stable for* (43) *then the origin is locally asymptotically stable for* (42).

Proof. If (43) has the origin asymptotically stable then, by Theorem 8.15, there exists a symmetric and positive definite matrix P such that the function $V(x) = \frac{1}{2}x \cdot Px$ satisfies, for every $x \neq 0$, the inequality

$$\nabla V(x) \cdot Ax < -rV(x).$$

Because $f(0) = 0$ and by the definition of differentiability of f at 0,

$$f(x) = \nabla f(x)x + o(x) = Ax + o(x),$$

where $\lim_{\|x\|\to 0} \frac{\|o(x)\|}{\|x\|} = 0$. Then

$$\begin{aligned}
\nabla V(x)\cdot f(x) &= \nabla V(x)\cdot(Ax + o(x)) < -rV(x) + (Px)\cdot o(x) \leq -rV(x) + \|Px\|\|o(x)\| \\
&\leq -\frac{1}{2}x\cdot Px + k\|x\|\|o(x)\|
\end{aligned}$$

where k is such that $\|Px\| \leq k\|x\|$ for all $x \in \mathbb{R}^n$. Then

$$\nabla V(x)\cdot f(x) \leq \|x\|^2\left(-\frac{1}{2}\frac{x}{\|x\|}\cdot P\frac{x}{\|x\|} + k\frac{\|o(x)\|}{\|x\|}\right) \leq \|x\|^2\left(-m + k\frac{\|o(x)\|}{\|x\|}\right),$$

where $m = \min_{\|y\|=1} V(y)$ is positive because V is continuous and positive for $y \neq 0$. There exists $\delta > 0$ such that, for all $x \neq 0$ with $\|x\| < \delta$, $\frac{\|o(x)\|}{\|x\|} < \frac{m}{2k}$. Then, for all such x,

$$\nabla V(x)\cdot f(x) < -\frac{m}{2}\|x\|^2 < 0.$$

Consequently, V is a local Lyapunov function for (42) and Theorem 8.9 concludes that the origin is locally asymptotically stable for (42). ◻

Example 8.17 Consider the differential equation (42) in $\mathbb{R}^2$ given by

$$f(x) = \begin{pmatrix} -x_1(1+4x_2^2) \\ x_2(2x_1^2 - 3 - e^{x_1}) \end{pmatrix}.$$

Then

$$\nabla f(x) = \begin{pmatrix} -1+4x_2^2 & -8x_1x_2 \\ x_2(4x_1 - e^{x_1}) & 2x_1^2 - 3 - e^{x_1} \end{pmatrix}, \quad \nabla f(0) = \begin{pmatrix} -1 & 0 \\ 0 & -4 \end{pmatrix},$$

the eigenvalues of $\nabla f(0)$ are -1 and -4 and hence, by Theorem 8.4, the origin is asymptotically stable for linearization (43). Theorem 8.16 now implies that the origin is locally asymptotically stable for the original nonlinear differential equation (42). Since local asymptotic stability entails Lyapunov stability, to verify if the origin is globally asymptotically stable for (42) one only needs to check if the origin is globally attractive. A rough argument suggesting that this is, in fact, true is as follows: $\dot{x}_1 = -x_1(1+4x_2^2)$, and since $1 + 4x_2^2 \geq 1$ for all x_2, x_1 should converge to 0 no slower than solutions to $\dot{x}_1 = -x_1$ do. This is valid independently of initial condition $x_1(0)$. Then, since for large t, $x_1(t)$ is small, the coefficient $2x_1^2 - 3 - e^{x_1}$ in $\dot{x}_2 = x_2(2x_1^2 - 3 - e^{x_1})$ is eventually negative, in fact less than -3. This ensures convergence of x_2 to 0. One could now conclude convergence to 0 of all solutions to (42), and so global attractivity of the origin, except for one gap in the argument: what happens to x_2 for small t, before the discussed coefficient is less than -3. For example, could it experience finite-time blow up? To this end, one should note that x_1^2 is decreasing, due to $\dot{x}_1 = -x_1(1+4x_2^2)$, and then $\dot{x}_2 = x_2(2x_1^2 - 3 - e^{x_1})$ ensures that x_2 does not grow faster than exponentially. Hence, finite-time blow up is impossible and the conclusion about convergence to 0 is correct.

Since the argument just presented, about convergence to 0 of all solutions to (42), may appear not convincing, one can try to check global asymptotic stability for (42) with a

Lyapunov function. Here, linearization is helpful too. The function $V(x) = \frac{1}{2}\|x\|^2$ is a Lyapunov function for (43) and it does not hurt to check if it is a Lyapunov function for (42). Note that the proof of Theorem 8.16 suggests that this V is a local Lyapunov function for (42), so it remains to check if it is a global one. One obtains

$$\begin{aligned}\nabla V(x) \cdot f(x) &= x_1\left(-x_1(1+4x_2^2)\right) + x_2\left(x_2(2x_1^2 - 3 - e^{x_1})\right) = -x_1^2 - 2x_1^2x_2^2 - 3x_2^2 - e^{-x_1}x_2^2 \\ &\le -x_1^2 - 3x_2^2\end{aligned}$$

and consequently V is a Lyapunov function for (42). Hence the origin is globally asymptotically stable for (42). △

The reverse implication in Theorem 8.16, that local asymptotic stability of the origin for (42) implies asymptotic stability for linearization (43), is false. To see this, it is enough to consider $\dot{x} = -x^3$ in $\mathbb{R}$. This nonlinear differential equation has 0 asymptotically stable. The linearization is $\dot{x} = 0$, and 0 is not attractive for this linear differential equation.

8.6 Perturbations and asymptotic stability

This section briefly discusses how perturbations of a differential equation influence asymptotic stability. Consider first a linear differential equation

$$\dot{x} = Ax \tag{44}$$

for which the origin is asymptotically stable. Then, for a matrix ΔA with sufficiently small entries, the linear differential equation

$$\dot{x} = (A + \Delta A)x$$

also has the origin asymptotically stable. One way to justify this is by considering eigenvalues. Eigenvalues depend continuously on the entries of the matrix. Consequently, if entries of ΔA are small, then eigenvalues of $A + \Delta A$ are close to eigenvalues of A. Now, asymptotic stability for (44) implies, by Theorem 8.4, that eigenvalues of A have negative real parts. Then, for small enough Δ, eigenvalues of $A + \Delta A$ have negative real parts, and consequently, the origin is asymptotically stable for $\dot{x} = (A+\Delta A)x$. Another way to justify this, using the methods developed in this chapter, is to rely on Lyapunov functions and their strict decrease, as in Theorem 8.15.

Problem 8.18 *For a 2×2 matrix A, show that a Lyapunov function $V(x) = x \cdot Px$ for* (44)*, as described in Theorem 8.15, is a Lyapunov function for $\dot{x} = (A + \Delta A)x$ if entries of ΔA are sufficiently small.*

Small linear perturbations can destroy asymptotic stability for nonlinear differential equations. For example, $\dot{x} = -x^3$ has the origin globally asymptotically stable but for an arbitrarily small $\varepsilon > 0$, this property fails for the differential equation $\dot{x} = -x^3 + \varepsilon x$. Indeed, for $x \in (0, \sqrt{\varepsilon})$, one has $\dot{x} > 0$ while for $x \in (-\sqrt{\varepsilon}, 0)$, one has $\dot{x} < 0$.

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