HYBRID SYSTEMS: GENERALIZED SOLUTIONS AND ROBUST STABILITY [1]

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Abstract: Robust asymptotic stability for hybrid systems is considered. For this purpose, a generalized solution concept is developed. The first step is to characterize a hybrid time domain that permits an efficient description of the convergence of a sequence of solutions. Graph convergence is used. Then a generalized solution definition is given that leads to continuity with respect to initial conditions and perturbations of the system data. This property enables new results on necessary conditions for asymptotic stability in hybrid systems.

Keywords: Hybrid systems, generalized solutions, set convergence, set-valued maps, asymptotic stability, Lyapunov functions, robustness.

1. INTRODUCTION

One of the most common tasks in nonlinear control is to design a feedback algorithm that robustly, asymptotically steers a dynamical system to a target set. This fact motivates the extensive literature on asymptotic stabilization for nonlinear differential equations and difference equations.

In a quest to provide more flexible tools for achieving the stabilization task and wider relevance for its solution, recent research efforts have focused on developing control algorithms that produce closed-loop systems where continuous variables interact with variables that make instantaneous jumps. A special case is when the control algorithm contains logic variables which take on discrete values. Systems with continuous variables and variables that jump are called hybrid systems.

In order to understand the capabilities of hybrid control systems, one of the first steps is to get a firm grasp on what is meant by a solution to a hybrid system. Some of the issues that come up when talking about solutions of hybrid systems include generalizing the "time" domain of a solution, characterizing when jumps happen, and dealing with systems that make an infinite number of jumps in a finite amount of ordinary time. Several different hybrid solution concepts have appeared in the literature. See, for example, (Tavernini, 1987), (Michel and Hu, 1999), (Lygeros *et al.*, 1999), (van der Schaft and Schumacher, 2000), (Aubin *et al.*, 2002), and (Prieur, 2003).

Together with the definition of solutions, it is also important to understand their structural properties, including continuity with respect to initial conditions and perturbations of the system data. Some work in this direction can be found in (Hiskens and Pai, 2000), (Simić *et al.*, 2001), (Prieur, 2003), (Prieur and Astolfi, 2003), (Lygeros *et al.*, 2003). Somewhat related is work on continuous dependence for impulsive differen-

[1] Partially supported by the Army Research Office under Grant no. DAAD19-03-1-0144, the National Science Foundation under Grant no. CCR-0311084 and Grant no. ECS-0324679, and by the Air Force Office of Scientific Research under Grant no. F49620-03-1-0203.

tial inclusions. See, for example, (Plotnikov and Kitanov, 2002).

The next step in understanding the stabilizing capabilities of hybrid control systems is to develop characterizations of asymptotic stability, which is typically induced for the purpose of steering to a target set asymptotically. This concept and its robustness properties are well-understood for differential and difference equations. Asymptotic stability has been addressed in the hybrid systems literature and many sufficient conditions have been provided. See (Branicky, 1998), (Ye *et al.*, 1998), (Michel and Hu, 1999), (DeCarlo *et al.*, 2000), (Liberzon, 2003), and (Lygeros *et al.*, 2003).

Certain necessary conditions for asymptotic stability, including some converse Lyapunov theorems, have been established as well. See, for example, (Ye *et al.*, 1998), (Michel and Hu, 1999). However, to the best of our knowledge, missing from the literature are general statements for hybrid systems about asymptotic stability's robustness, which is important for predicting the behavior of a hybrid control system in the presence of measurement noise and other modeling uncertainty. For this reason, when robust asymptotic stability is desired, robustness is usually proved separately. An example of this situation can be found in (Prieur and Astolfi, 2003).

General results on robust asymptotic stability for hybrid systems probably are not available because the efforts to develop solution notions for hybrid systems rarely have been linked to the pursuit of robustness for asymptotic stability. A notable exception can be found in (Prieur, 2003).

In this paper, we will attempt to motivate giving attention to this link between the definition of solution and robust asymptotic stability. To that end, we will sketch a notion of solution for hybrid systems that bears some resemblance to notions that have appeared previously but which has some unique features. For "single-valued" hybrid systems, it can be thought of as a "generalized" hybrid solution notion, a la Filippov (Filippov, 1960) (see also (Filippov, 1988)) or Krasovskii (Krasovskii, 1970) for discontinuous differential equations. Like for differential equations, hybrid Filippov and Krasovskii solutions have strong structural properties, including nice continuity properties with respect to perturbations of initial conditions and system data. In turn, asymptotic stability using hybrid Krasovskii solutions can be linked to robustness to measurement noise and other small perturbations in hybrid control systems, like in differential equations (Hermes, 1967), (Hájek, 1979). Moreover, in contrast to the case for differential equations, the use of hybrid Filippov or Krasovskii solutions does not preclude nice closed-loop behaviors resulting from hybrid controllers that are not possible using their continuous-time counterparts. See (Ceragioli, 2002) for examples illustrating the negative consequences of using Filippov or Krasovskii solutions for certain continuous-time control systems that use discontinuous feedback. We will conclude with a series of necessary statements that follow from asymptotic stability using the solution notion presented in this paper.

2. EXAMPLES OF HYBRID SYSTEMS

In a hybrid dynamical system, the state sometimes flows (continuously) while at other times it makes jumps. Whether flow occurs or a jump occurs depends on the state's location in the state space. Thus, a hybrid dynamical system is usually described by two functions, f and g, and two sets C and D. The function f generates a differential equation that governs flow while the function g generates a reset equation that governs jumps. The function f is often only specified for variables that can flow while the function g is often only specified for variables that can jump. The set C indicates where in the state space flow may occur while the set D indicates where in the state space jumps may occur. Where these sets overlap, both flowing and jumping may be possible. Some examples of hybrid systems are given next. We will not be precise about what is meant by a solution for these systems until subsequent sections.

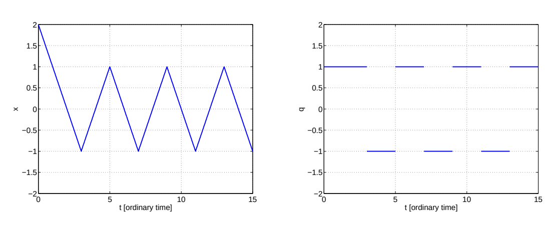
Oscillator using hysteresis. This example describes an oscillator with a single continuous state, denoted x, and a state q that can take values in the discrete set $\{-1, 1\}$. Thus, the state space for (x, q) is $\mathbb{R} \times \{-1, 1\}$. The flow equation for x is given by $\dot{x} = q =: f(q, x)$ and it applies on the set

$$C := \{q = -1 \ \& \ x > -1 \ , \text{ or } q = 1 \ \& \ x < 1\}.$$

The jump equation is $q^+ = -q$, applying on

$$D := \{q = -1 \ \& \ x \leq -1 \ , \text{or } q = 1 \ \& \ x \geq 1\}.$$

A sample trajectory is shown in Figure 1. The trajectories tend to the set $[-1, 1] \times \{-1, 1\}$.



(a) continuous state (b) discrete state

Fig. 1. *Trajectories for oscillator using hysteresis.*

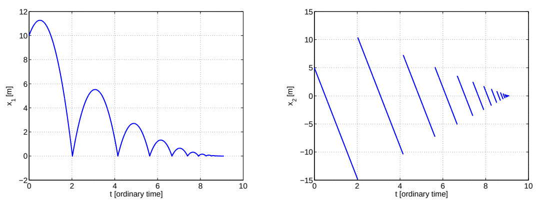
Bouncing ball. This example describes the behavior of a ball bouncing on the floor. The state is two-dimensional, with x_1 being the ball's height above the floor and x_2 being the ball's velocity. The state space corresponds to $x_1 \geq 0$ and $x_2 \in \mathbb{R}$. The flow equation is given by

$$\dot{x} = \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} =: f(x) \ ,$$

where $-\gamma$ is the acceleration due to gravity. The flow equation applies in

$$C := \{x_1 > 0, \ \text{or} \ x_1 = 0 \ \& \ x_2 > 0\} \, .$$

The jump equation for velocity is $x_2^+ = -\mu x_2 =: g(x)$ where $\mu \in (0, 1)$ is a dissipation factor. It applies in $D := \{x_1 = 0 \ \& \ x_2 \leq 0\}$. Figure 2 contains a sample trajectory for this system. Visually, the trajectories tend toward the origin.



(a) Height (b) Velocity

Fig. 2. *Bouncing Ball trajectories.*

Nonholonomic integrator. Consider a simplified version of the hybrid controller for a nonholonomic integrator in (Hespanha and Morse, 1999) proposed by (Ryan, 1996). Let $q \in \{0, 1\}$, $x \in \mathbb{R}^3$. The flow is governed by $\dot{x} = f(x, q)$ where $f(x, q) := (u_1, u_2, x_1 u_2 - x_2 u_1)$ and

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = (1 - q) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + q \begin{bmatrix} -x_1 + \dfrac{2x_2 x_3}{x_1^2 + x_2^2} \\ -x_2 - \dfrac{2x_1 x_3}{x_1^2 + x_2^2} \end{bmatrix} \, .$$

Additionally, $f(0, 1) = 0$. It applies on the set

$$C := \begin{array}{l} \{q = 0, \ x_1^2 + x_2^2 \leq \rho |x_3|\} \\ \quad \cup \ \{q = 1, \ x_1^2 + x_2^2 \geq |x_3|\} \, , \end{array}$$

where $\rho > 1$. The jump equation for q is

$$q^+ = 1 - q =: g(x, q)$$

which applies on D, the set of points with $x \neq 0$ in

$$\begin{array}{l} \{q = 0, \ x_1^2 + x_2^2 \geq \rho |x_3|\} \\ \quad \cup \ \{q = 1, \ x_1^2 + x_2^2 \leq |x_3|\} \, . \end{array}$$

The trajectories of this hybrid system tend to the set where $x = 0$ and either $q = 0$ or $q = 1$. Indeed, if initially $q = 0$, eventually there is a jump to $q = 1$ and from then on, q is constant with x converging to 0 exponentially fast.

Sampled-data stabilization of an integrator to the integers. Consider stabilizing the state of an integrator $\dot{x} = u$ to the set of integers using sampled-data feedback. Let $T > 0$ denote the period between taking samples of the state. Let $\text{int}(x)$ denote the closest integer to the value $x \in \mathbb{R}$. For the sake of resolving ambiguity for values x halfway between two integers, define $\text{int}(x)$ to be the smaller of the two closest integers in this case. The state of the overall system will consist of the state x of the integrator, the sampled feedback value, denoted z, and ρ which keeps track of time. The flow equation can be written

$$\begin{aligned} \dot{x} &= T^{-1} z \\ \dot{z} &= 0 \\ \dot{\rho} &= T^{-1} \end{aligned}$$

and it applies on the set $C := \{\rho \leq 1\}$. The jump equation is given by

$$\begin{aligned} z^+ &= -x + \text{int}(x) \\ \rho^+ &= 0 \end{aligned}$$

and it applies on the set $D := \{\rho \geq 1\}$. The trajectories of this system tend to the set where x is an integer, $z = 0$ and $\rho \in [0, 1]$.

Rotate and dissipate. Consider a planar system with flow governed by the equation

$$\dot{x} = f(x) := \begin{bmatrix} x_2 \\ -x_1 \end{bmatrix} \ ,$$

which applies on the set

$$C := \{x \mid x_1^2 + x_2^2 \leq 2, x_1 \neq 0\},$$

and jumps governed by the equation

$$x^+ = g(x) := \begin{bmatrix} \mu x_2 \\ 0 \end{bmatrix} \qquad \mu \in (0, 1) \ ,$$

which applies on the set

$$D := \{x \mid x_1^2 + x_2^2 \leq 1, x_1 = 0\}.$$

Intuitively, the trajectories that originate in the unit disk rotate clockwise continuously until they hit the x_2-axis at which point they rotate instantaneously to the x_1-axis, decreasing in magnitude by the factor μ. This process is recurrent and, in this way, these trajectories tend to the origin. Outside of the unit disk the trajectories rotate clockwise continuously without jumps or dissipation.

3. HYBRID TIME DOMAINS

In this section we make more precise what we mean by a solution for a hybrid dynamical system. The first issue to address is the "time" domain on which the solutions are defined.

To ease notation, from now on when making general statements about hybrid systems, we do not mention the discrete variable q explicitly, and denote the state of the system as just x. The discrete set of potential values of q – often consisting of descriptive elements like "on" and "off" – can be identified with a finite subset of integers, and then one of the coordinates of x can be used to represent q.

3.1 Partitioning ordinary time

The simplest choice is to use the ordinary time domain broken into non-overlapping intervals that cover the ordinary time domain. Then, the solutions to the hybrid system are functions [2] satisfying

$$\dot{x}(t) = f(x(t)) \tag{1}$$

and $x(t) \in C$ on each interval of continuity and whose right limits $x(t^+) = \lim_{\tau \searrow t} x(\tau)$ at the jump times t, which are determined by $x(t) \in D$, are related to $x(t)$ through

$$x(t^+) = g(x(t)) \; . \tag{2}$$

This choice of the time domain does not allow for more than one jump at a given time. This is not an obstacle to the analysis of systems for which times between jumps can be uniformly bounded below (as it is the case for the Oscillator with Hysteresis and for the Nonholonomic Integrator). However, it may preclude limits of solutions with increasing number of jumps in a finite amount of time.

For example, consider the Bouncing Ball. There does not exist a solution originating at the origin: the "expected" solution $x(t) = 0$ for all $t \geq 0$ does not satisfy the continuous dynamics (1), while a solution that keeps on jumping according to (2) from the origin to the origin is not piecewise continuous. For any initial point above the floor, there does exist a unique solution, and each such solution is *Zeno*: there are infinitely many jumps in a finite amount of time (as illustrated on Figure 2). Note, however, that the solutions that one obtains by dropping the ball from lower and lower heights do not have a "limiting" solution.

[2] Throughout the paper, when the time derivative of a function is used, the function is assumed to be piecewise absolutely continuous and the derivative condition is assumed to hold almost everywhere.

Lack of "limiting" solutions is a fundamental obstacle to robustness analysis, as one cannot expect a reasonable dependence of solutions on initial conditions and perturbations.

3.2 Hybrid time domain

An alternate approach is to consider the state of a system not only as a function of time but also of the number of jumps that occurred. To this end, by a *hybrid time domain* we understand a subset of $[0, +\infty) \times \mathbb{N}_0$ given as a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$, where the numbers $0 = t_0, t_1, ...$ form a finite or infinite and nondecreasing sequence. Here, $\mathbb{N}_0 = \{0, 1, 2, ...\}$. We do allow for the "last" interval to be of the form $[t_j, T) \times \{j\}$ with T finite or $T = +\infty$.

A sketch of a hybrid time domain corresponding to the Bouncing Ball is shown in Figure 3.

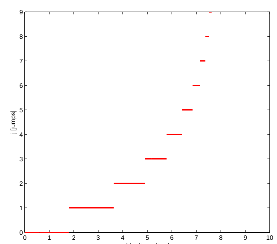


Fig. 3. *Hybrid time domain for the Bouncing Ball.*

This concept builds upon the one in (Lygeros *et al.*, 1999), (Aubin *et al.*, 2002), and (Lygeros *et al.*, 2003), but gives a more explicit role to the "discrete" variable j: the state of the system will be parameterized by (t, j). (Height of the Bouncing Ball as a function of (t, j) is shown in Figure 4; cf. Figure 2.) The benefits of this approach will be underlined in the sections to come. We add that each hybrid time domain can be embedded in $[0, +\infty)$ via an "order-preserving" function $(t, j) \to t + j$ (embeddings of hybrid systems into $[0, +\infty)$ are relied upon in (Michel and Hu, 1999), (Michel, 1999); here we note that different hybrid time domains need not be subsets of a pre-described "time-space").

In this framework, a solution to the hybrid system is a function defined on a hybrid time domain, such that

$$\dot{x}(t, j) = f(x(t, j)) \tag{3}$$

and $x(t, j) \in C$ are satisfied on (t_j, t_{j+1}), while for all j,

$$x(t_{j+1}, j + 1) = g(x(t_{j+1}, j)) \tag{4}$$

and $x(t_{j+1}, j) \in D$.

Note that unless one makes *a priori* requirements that $t_{j+1} > t_j$, more than one jump can occur at a single moment of ordinary time. (Figure 5

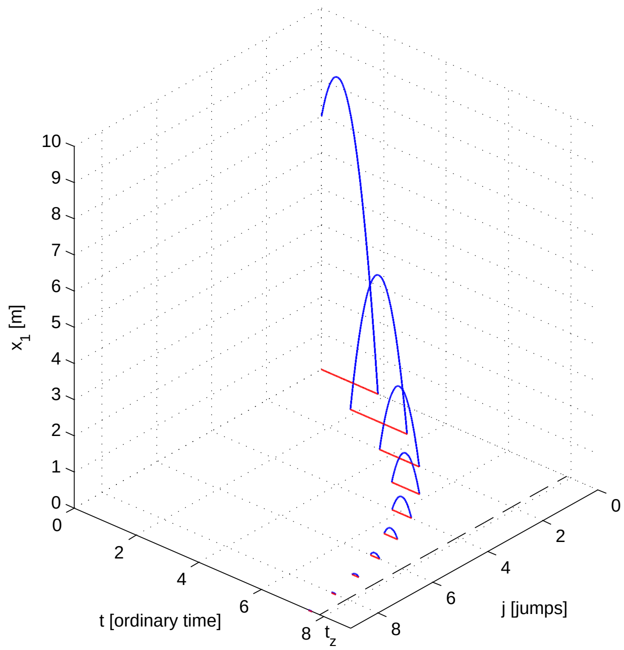


Fig. 4. *Height of the Bouncing Ball as a function of* (t, j) *on its hybrid time domain.*

shows a hybrid time domain with three jumps at a single moment, this corresponds to the behavior of Newton's cradle with four balls, see (van der Schaft and Schumacher, 2000).) In fact, a purely discrete evolution is possible – if $t_j = 0$ for all j, then the hybrid time domain can be identified with $1, 2, ...$ and thus the evolution is according to a difference equation (4). Still, further generalization is possible, to allow multiple ordinary times at which an infinite number of jumps occurs; see (van der Schaft and Schumacher, 2000). We do not pursue this here.

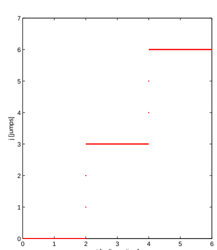


Fig. 5. *Hybrid time domain for Newtons cradle.*

With the new concept of a hybrid time domain, there does exist a solution originating at the origin for the Bouncing Ball. It is a purely "discrete" solution – it keeps jumping from the origin to the origin; the differential equation (3) does not come into play. As we will see later, this is in fact the previously missing "limiting" solution.

The question of existence of solutions to a hybrid system now reduces to the separate questions for (3) and (4). That is, there exists a solution to the hybrid system originating at x^0 if either

- $x^0 \in D$ and a jump is possible (i.e. $g(x_0) \in C \cup D$), or
- $x^0 \in C$ and a continuous flow is possible (conditions for this involve viability theory, see (Aubin *et al.*, 2002)).

Previously, one also had to check whether the jump occurs to a state from which a continuous flow is possible.

The symmetric treatment of t and j yields a more natural view of Zeno solutions. We say that a solution x to the hybrid system is *complete* if its hybrid time domain is unbounded. If x is complete but the domain is bounded "in the j-direction", that is its projection onto $\mathbb{N}_0$ is bounded, then the last interval in the domain is of the form $[t_j, \infty) \times \{j\}$. On the other hand, complete solutions that have the domain bounded "in the t-direction" are exactly Zeno solutions.

Note though that changing the time domain does not address all of the issues with "limiting" solutions. Indeed, for the Rotate and Dissipate example, solutions originating at $(\epsilon, 0)$ have the same hybrid time domains which are unbounded in both t and j directions (and so are not Zeno), and in a sense, converge uniformly to a function that is equal to zero. The latter is not a solution. It is also the regularity of C, D, f, and g that plays a role in a satisfactory "limiting" process. We pursue this further in Section 6.

4. CONTINUITY OF SOLUTIONS: WHAT IS DESIRED AND HOW IT MIGHT FAIL

A key reason for recognizing j as an independent variable, equally important as t, is that the otherwise cumbersome or limiting concepts of closeness and convergence of solutions can be treated globally, in particular through the well-established nonsmooth analysis concept of graphical convergence. The classical, measured pointwise, distance is not very well-suited for hybrid trajectories, as it does not handle jumps well. Resulting concepts of closeness often require that jumps occur at exactly the same times. Hybrid time seemingly adds another challenge, of having trajectories defined on different sets. Focusing on graphs of trajectories, and discussing their convergence as sets, clears the way to overcoming these, and other obstacles.

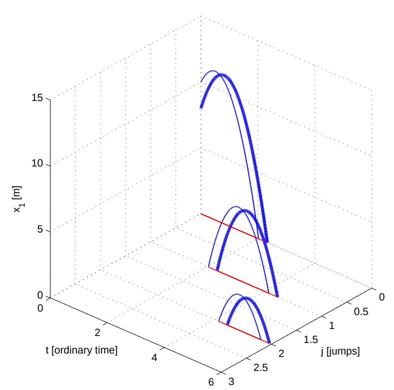


Fig. 6. *Bouncing ball: heights of "close" solutions.*

Given a hybrid trajectory x with domain $\operatorname{dom} x$, its *graph* is the set in $[0, +\infty) \times \mathbb{N}_0 \times \mathbb{R}^n$ given by

$$\operatorname{gph} x := \{(t, j, x(t, j)) \mid (t, j) \in \operatorname{dom} x\}.$$

A sequence of trajectories *converges graphically* if the corresponding graphs converge as sets. The corresponding concept of closeness is similar to the one in the definition of continuous hybrid systems in (Lygeros *et al.*, 2003). Two hybrid trajectories x, x' with domains $\operatorname{dom} x$, $\operatorname{dom} x'$ are ϵ-close on $[0, T] \times \{0, 1, \ldots, J\}$ if

- for all $(t, j) \in \operatorname{dom} x$ such that $t \le T$ and $j \le J$, there exists s such that $(s, j) \in \operatorname{dom} x'$, $|t - s| < \epsilon$, and
$$\|x(t, j) - x'(s, j)\| < \epsilon,$$
- for all $(t, j) \in \operatorname{dom} x'$ such that $t \le T$ and $j \le J$ there exists s such that $(s, j) \in \operatorname{dom} x$, $|t - s| < \epsilon$, and
$$\|x'(t, j) - x(s, j)\| < \epsilon.$$

Similar conditions can be used to formalize graphical convergence of hybrid trajectories.

Vaguely, conditions for ϵ-closeness require that if no jumps occur near t for either trajectory, then the same number of jumps occurred already for each of them, and the trajectories should be close in the classical pointwise sense. If there is a jump at t for one trajectory, then there is a jump for the other at a nearby time. Note however that the jumps for x and x' need not occur at the same time. (This can be observed in Figure 6.) Furthermore, trajectories with continuous and discrete evolution can be close to one with only discrete behavior.

Convergence of sets in general, and convergence of graphs in particular, is a concept well-developed and often used in set-valued and nonsmooth analysis; see (Rockafellar and Wets, 1998), or (Aubin and Cellina, 1984) for applications in differential inclusion theory. One of the general results on set convergence implies that for essentially any sequence of solutions to a hybrid system, there exists a subsequence that converges graphically. Whether the limit is a solution to the hybrid system – a property fundamental in analyzing robustness – hinges upon the properties of the system data f, g, C, and D.

When discussing robustness of asymptotic stability, we will need to consider perturbed hybrid systems. For a hybrid system with initial conditions in a set K that has flow equation $\dot{x} = f(x)$, which applies when $x \in C$, and a jump equation $x^+ = g(x)$ which applies when $x \in D$, and for $\delta > 0$, a δ-perturbed solution will be a solution for the hybrid system that has initial conditions in a δ-neighborhood of K, has flow equation

$$\dot{x} = f(x + d_1) + d_2$$

which applies when $x + d_3 \in C$ and that has jump equation

$$x^+ = g(x + d_4) + d_5$$

which applies when $x + d_6 \in D$, where the disturbances d_i are possibly time-varying and are bounded in norm by δ on their domain.

The closeness we are after is that for each compact set of initial points, each $\epsilon > 0$ and each $T > 0$ and $J > 0$, there exists $\delta > 0$ such that each δ-perturbed solution is ϵ-close on $[0, T] \times \{0, 1, \ldots, J\}$ to some solution of the original system. At times, we will only ask for this property for disturbance signals that keep the hybrid system within its original state space. This is mainly an issue for systems that do not have an open state space.

We now discuss some of the previous examples with respect to the desired δ-perturbed ϵ-closeness.

Bouncing Ball, revisited. Recall that the bouncing ball system did not have a solution in the sense of Section 3.1 originating at $x = 0$. In the sense of Section 3.2, a solution does exist. It is given by $x(0, j) = 0$, with the domain $\mathrm{dom}\, x = \{0\} \times I\!N_0$.

Considering such a solution is essentially dictated by the goal of obtaining a successful convergence concept. Indeed, consider the ball being dropped from lower and lower height. A sample of resulting solutions is shown in Figure 7. The decreasing Zeno times are indicated as t_{zi} in the figure. The graphical limit of such solutions is exactly $x(0, j) = 0$ for all $j \in I\!N_0$. In other words, the solutions originating from $(\delta, 0)$ are close to the limiting one.

Sampled-data stabilization of an integrator to the integers, revisited. Suppose that ρ starts at one and z starts at zero. Consider the solution with x starting at $1/2$ compared to x starting at $1/2+\delta$ with $\delta \in (0, 1/2]$. In either case, a jump occurs immediately, and after the jump z equals $-1/2$ in the first case and $1/2 - \delta$ in the second case. The next jump does not occur until $t = T$. In the first case, after T seconds $x = 0$. In the second case, no matter how small $\delta > 0$ is, after T seconds $x = 1$. The perturbed solution is not close to the unperturbed solution.

Rotate and dissipate, revisited. First consider the solution concept of Section 3.1, and the unique solution originating from $(1, 0)$. As previously outlined, it tends to the origin. However, the (also unique) solution from $(1 + \delta, 0)$ is periodic, and have constant norm. They are not close to that from $(1, 0)$. Note that discarding the endpoints $(0, \pm 1)$ from the jump set D is not a remedy – then, the solutions from $(1 - \delta, 0)$ are not close to the periodic one from $(1, 0)$.

Considering the solution concept of Section 3.2 does not alter the situation just described.

So far we only varied the initial condition. Similar issues arise under other perturbations. For example, arbitrarily small d_3 and d_6 can cause the solutions from $(\alpha, 0)$ with $\alpha \in (-1, 1)$ to be periodic. This happens since the jump set D is too "thin".

Because of some of the issues that come up in these examples, like nonexistence and/or lack of continuity with respect to initial conditions and/or perturbations, we wish to explore generalized notions of solutions that don't suffer these difficulties. Such generalized notions of solutions will parallel what has been done in (Filippov, 1960) and (Krasovskii, 1970) for discontinuous differential equations. Similar generalized notions of solution have also been used for discontinuous discrete-time systems in (Kellett and Teel, 2004). As we point out in the next section, the advantage of these generalized notions of solution is that they have nice structural properties. The disadvantage, especially in continuous time, is that they preclude certain behaviors. We will argue later that a generalized notion of solution for hybrid systems can be given that has nice structural properties and does not preclude the behaviors that hybrid controllers have been relied upon to produce in certain control systems.

5. GENERALIZED SOLUTIONS FOR DIFFERENTIAL AND DIFFERENCE EQUATIONS

We start by considering the relation between solution concepts and robust asymptotic stability for differential and difference equations. This topic has received significant attention recently, for example in the papers (Clarke *et al.*, 1997), (Ledyaev and Sontag, 1999), (Sontag, 1999), (Ancona and Bressan, 1999), (Clarke *et al.*, 2000), (Ceragioli, 2002), (Ancona and Bressan, 2003), (Clarke and Stern, 2003), (Kellett *et al.*, 2004), (Kellett and Teel, 2004), (Grimm *et al.*, 2004).

5.1 Differential equations

Consider steering the state of an integrator $\dot{x} = u$ to the set of integers using the feedback

$$u = -\mathrm{sgn}(x - \mathrm{int}(x)) =: \sigma(x)$$

where "int" was defined earlier when using sampled-data feedback for this problem, and $\mathrm{sgn}(0) = 0$.

5.1.1. Global asymptotic stability w/o robustness

First, by solution to the closed-loop system we will mean any function $x(\cdot)$ satisfying $\dot{x}(t) = \sigma(x(t))$. A similar definition will be used for the corresponding system with measurement noise $\dot{x}(t) = \sigma(x(t) + e(t))$. With this definition, the solutions starting halfway in between neighboring integers are not unique. However, the solutions are such that the set of integers is globally asymptotically stable for the system without measurement noise. In particular, the distance to the set of integers is never bigger than the initial distance and the trajectories converge to the integers in no more than one half of a second.

Unfortunately, this global asymptotic stability is not robust to measurements errors, no matter how

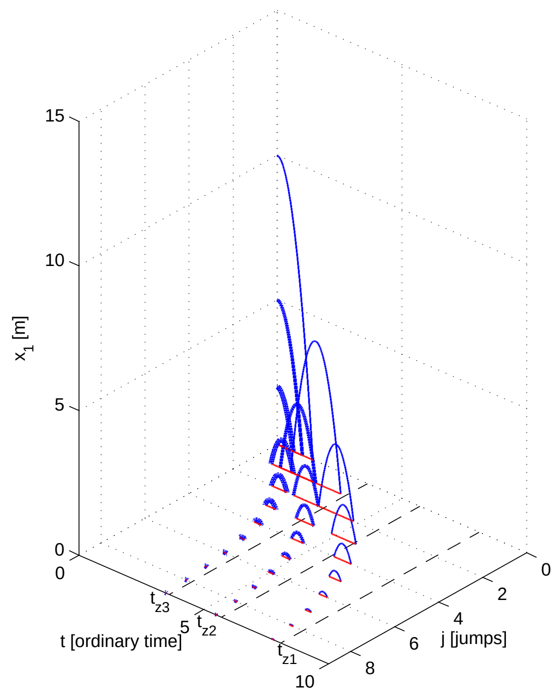


Fig. 7. *Bouncing ball: "converging" solutions.*

small they are. In particular, for initial conditions halfway between integers, the solution

$$x(0) + \varepsilon \cdot \mathrm{tri}(t/\varepsilon) \ ,$$

where "tri" denotes the triangle wave given by

$$\mathrm{tri}(s) = (-1)^i(s-2i) \quad \forall s \in [2i-1, 2i+1] \ , \ i \in \mathbb{N}_0$$

is induced by the measurement noise

$$e(t) = -\varepsilon \cdot \mathrm{tri}(t/\varepsilon) + \varepsilon \cdot \mathrm{tri}(1 + t/\varepsilon)$$

where $\varepsilon > 0$ can be arbitrarily small. Therefore, in the presence of arbitrarily small measurement noise, the trajectories may not get close to the set of integers. The global asymptotic stability property for the unperturbed system provides no indication of the potential problems with measurement noise near the points halfway between integers.

5.1.2. Robustness w/o global asymptotic stability Now by a solution we mean any function $x(\cdot)$ satisfying $\dot{x} \in S(x(t))$ where $S(x)$ is the interval $[-1, 1]$ at the integers and also at the points halfway between integers, and $S(x) = \sigma(x)$ at all other points. This is the generalized solution concept for discontinuous differential equations, due to Filippov (and Krasovskii). See, for example, (Filippov, 1988). Due to the change in the solution definition, the set of integers is no longer globally asymptotically stable. Instead it is locally asymptotically stable with basin of attraction equal to all points on the real line except those halfway between neighboring integers. This is the downside of the generalized solution concept. The upside is that this stability property is robust. The boundary of the basin of attraction indicates exactly where the possibility of problems with measurement noise exists. Away from these points, small measurement noise cannot keep the trajectories very far from the integers.

At this point, it is worth noting that the set of integers is made globally asymptotically stable in a manner that is robust to measurement noise when using sampled-data feedback as discussed earlier. This can be seen directly, but also follows from general results coming in later sections.

5.2 Difference equations

A similar phenomenon can occur for discrete-time systems. Consider the control system

$$x^+ = g(x, u) := \begin{bmatrix} x_1(1-u) \\ |x|u \end{bmatrix} \qquad (5)$$

and the choice of state feedback

$$u = \kappa(x) := \begin{cases} 1 & \text{if } x_1 \neq 0 \\ 0 & \text{if } x_1 = 0 \ . \end{cases}$$

This control law actually results from applying model predictive control to (5) using the origin as a terminal constraint and a horizon of length two. See (Grimm *et al.*, 2004) for details.

5.2.1. Global asymptotic stability w/o robustness The first notion of solution that we consider is seemingly the most natural one. Namely, any sequence x_k that satisfies $x_{k+1} = g(x_k, \kappa(x_k))$ is a solution. With this definition, the solutions are unique and the origin is globally asymptotically stable. Indeed, each solution's distance from the origin is never bigger than the initial distance and each solution converges to the origin in two steps. However, this asymptotic stability is not robust to measurement noise. In particular, the feedback $u = \kappa(x + e)$ where e is an arbitrarily small but nonzero constant, yields the solution

$$\begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}, \begin{bmatrix} 0 \\ \|x_0\| \end{bmatrix}, \begin{bmatrix} 0 \\ \|x_0\| \end{bmatrix}, \ldots \qquad .$$

Again, global asymptotic stability for the unperturbed system provides no indication of potential problems with arbitrarily small measurement noise, this time along the positive x_2 axis.

5.2.2. Robustness w/o global asymptotic stability Now a generalized definition of solution is considered. It may not seem as natural, but it is effective at revealing the possibility of problems due to measurement noise and other types of perturbations. For more details see (Kellett and Teel, 2004). This time by solution we mean any sequence x_k that satisfies $x_{k+1} \in G(x_k)$ where $G(x) = g(x, \kappa(x))$ at all points except those on the positive x_2 axis where $G(x)$ is the two point set containing x and the origin. This is the set of all limiting vectors obtained by considering $g(y, \kappa(y))$ at points y arbitrarily close to x. (The construction of G is Filippov- and Krasovskii-like but G doesn't need to be convex in the discrete-time case.) Solutions in this new sense are not unique. Indeed, each solution moves to the positive x_2 axis in one step, if it wasn't there already, then can remain stationary on the x_2 for an arbitrary number of steps (or forever), and then may jump to the origin. The downside of using this generalized solution concept is that the origin is no longer asymptotically stable. It is only stable. The upside is that it accurately reflects what can occur with arbitrarily small measurement noise.

6. GENERALIZED SOLUTIONS FOR HYBRID SYSTEMS

Motivated by the desire for a solution concept that ensures asymptotic stability is robust, we consider solutions in a generalized sense for a hybrid system described by functions f, g and sets C, D. These solutions will be defined through hybrid time domains, and through regularizations of the system data f, g, C and D. Various regularizations are possible. The simplest regularization for the sets C and D comes by taking their closures $\overline{C}$ and $\overline{D}$. For f and g, we will work with set-valued extensions as hinted at in Section 5.

For a function f on C, its Krasovskii extension is a (possibly set-valued) mapping F_K on $\overline{C}$ such that $F_K(x)$ is the smallest closed convex set containing all limits of $f(x')$ as $x' \to x$. The Krasovskii extension of g on D to G_K on $\overline{D}$ is defined similarly, but without requiring convexity of $G_K(x)$. Filippov extensions are slightly more technical; vaguely, for piecewise continuous functions, the limits are taken only over points x' where f is continuous. In several cases, the Filippov and Krasovskii extensions are equal. This occurs for example when C is the union of finitely many regions C_i such that each C_i is a subset of the closure of its interior, and $f|_{C_i}$ is continuous. In general, Filippov extensions are smaller than Krasovskii extensions.

Letting F and G be the extensions of f and g in the Filippov or Krasovskii sense, when f and g are locally bounded, and for Filippov extensions also measurable, then the regularized data $(F, G, \overline{C}, \overline{D})$ satisfy the following *basic conditions:*

- sets $\overline{C}$ and $\overline{D}$ are closed,
- the mapping F has convex values, is locally bounded and has a closed graph,
- the mapping G is locally bounded and has a closed graph.

In general, we can consider set-valued hybrid systems with data that satisfies these basic conditions and make the following solution definition:

Recall that a hybrid time domain was a union of finitely or infinitely many intervals $[t_j, t_{j+1}] \times \{j\}$ in $[0, +\infty) \times \{0, 1, 2, ...\}$, with the last interval possibly open and unbounded. A *(generalized) solution* is a function x defined on a hybrid time domain $\operatorname{dom} x$ and such that

(S1) on each interval $[t_j, t_{j+1}] \times \{j\} \subset \operatorname{dom} x$ of positive length (so that $t_j < t_{j+1}$) we have

$$x(t, j) \in \overline{C}, \quad \text{and} \quad \dot{x}(t, j) \in F(x(t, j)),$$

(S2) for each $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$, we have

$$x(t, j) \in \overline{D}, \quad \text{and} \quad x(t, j + 1) \in G(x(t, j)).$$

The hybrid basic conditions on $(F, G, \overline{C}, \overline{D})$ lead to the desired convergence and ϵ-closeness for δ-perturbation results for the solutions.

We now discuss generalized solutions for some of the examples presented earlier.

Oscillator using hysteresis re-revisited. Here, the only change from the original data is

$$\overline{C} = \{q = -1 \;\&\; x \geq -1, \text{ or } q = 1 \;\&\; x \leq 1\}.$$

Unique generalized trajectories exist from every initial point in $\overline{C} \cup \overline{D} = \mathbb{R} \times \{-1, 1\}$, and the x-coordinate tends to $[-1, 1]$.

Bouncing ball re-revisited. Here, $\overline{C} = \{x \mid x_1 \geq 0\}$, and the other data does not change, that is $F = f$, $G = g$, $\overline{D} = D$. This by itself does not change the behavior of the system; rather, it is passing to a hybrid time domain that allows for the important "limiting" solution originating from the origin, as noted in Section 4.

Sampled-data stabilization of an integrator to the integers, revisited. Here $\overline{C} = C$, $\overline{D} = D$, $F = f$ and

$$G(x, z, \rho) = \begin{bmatrix} -x + \mathrm{INT}(x) \\ 0 \end{bmatrix}$$

where $\mathrm{INT}(x) = \mathrm{int}(x)$ except at points halfway between integers where $\mathrm{INT}(x)$ contains both closest integers to x. Since the discontinuous function is in the jump equation rather than the flow equation, it is not necessary to take the convex hull. Generalized solutions starting from $\rho = 1$, and x halfway between integers are not unique. Still, they all tend to the set where x is an integer, $z = 0$ and $\rho \in [0, 1]$.

Rotate and dissipate re-revisited. Here, the only change is $\overline{C} = \left\{x \mid x_1^2 + x_2^2 \leq 1\right\}$. The solutions to this system are no longer unique, and do not tend to 0, as noted in Section 4. This can be related to lack of robustness in the original formulation. However, the issue of convergence of solutions from $(1 + \delta, 0)$ to a solution from $(1, 0)$ is resolved.

We did not discuss continuity with respect to perturbations of the system data directly for these examples. This property is addressed indirectly through the results of the next section, at least for those systems that have an asymptotically stable compact set. This is the case for each example except the rotate and dissipate system.

7. CONSEQUENCE: ROBUST STABILITY

Our motivation for revisiting the solution concept for hybrid systems was to find a solution notion that provided robustness of asymptotic stability for free, without destroying the unique features of hybrid systems in the process. For this purpose, we introduced generalized hybrid solutions.

This solution notion enables establishing several interesting results related to asymptotic stability for hybrid systems, even those that are set-valued from the beginning. The results parallel what is known to hold for differential and difference equations and inclusions. See, for example, (Teel and Praly, 2000) in continuous time and (Kellett and Teel, 2004), (Kellett, 2002) in discrete time. The details of the results mentioned below will be given elsewhere.

Throughout this section, we assume that the data of the hybrid system, $(F, G, \overline{C}, \overline{D})$, satisfies the basic conditions given in the previous section. Now we define asymptotic stability and robust asymptotic stability for such systems.

For a hybrid system $(F, G, \overline{C}, \overline{D})$, a compact set $\mathcal{A}$ is said to be locally asymptotically stable (LAS) relative to a set $\mathcal{X} \supseteq \mathcal{A}$ if

- (Stability) for each $\epsilon > 0$ there exists $\delta > 0$ such that for each initial condition in $\mathcal{X}$ that is in a δ-neighborhood of $\mathcal{A}$, each (generalized) solution has an unbounded domain and throughout its domain the solution stays in $\mathcal{X}$ and in an ϵ-neighborhood of $\mathcal{A}$,
- (Attractivity) the set of points in $\mathcal{X}$ such that:

  each (generalized) solution has an unbounded domain, is contained in $\mathcal{X}$, and converges to $\mathcal{A}$

  contains a neighborhood of $\mathcal{A}$ intersected with $\mathcal{X}$.

We use $\mathcal{R}$ to denote the set of points in $\mathcal{X}$ attracted to $\mathcal{A}$ and call this set the basin of attraction for $\mathcal{A}$.

When the state-space domain $\mathcal{X}$ is open, $\mathcal{X}$ can be omitted completely from the characterization above. When it is omitted, we implicitly mean that $\mathcal{X}$ is open. When $\mathcal{X}$ is not open, it is often possible to extend the hybrid system to one defined on an open set containing $\mathcal{X}$ without changing the solutions that start in $\mathcal{X}$ and preserving stability and attractivity. This is useful to know for the characterization of robustness given below.

For a hybrid system $(F, G, \overline{C}, \overline{D})$ a compact set $\mathcal{A}$ that is locally asymptotically stable with (open; see below) basin of attraction $\mathcal{R}$ is said to be robustly asymptotically stable if there exists a continuous function $\delta : \mathcal{R} \to \mathbb{R}_{\geq 0}$, positive definite with respect to $\mathcal{A}$, such that for the hybrid system $(F_\delta, G_\delta, \overline{C}_\delta, \overline{D}_\delta)$ the set $\mathcal{A}$ is locally asymptotically stable with basin of attraction $\mathcal{R}$, where the hybrid system $(F_\delta, G_\delta, \overline{C}_\delta, \overline{D}_\delta)$ is defined as:

$$\begin{aligned} F_\delta(x) &:= \overline{\mathrm{co}} F(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \ , \\ G_\delta(x) &:= G(x + \delta(x)\overline{\mathcal{B}}) + \delta(x)\overline{\mathcal{B}} \ , \\ \overline{C}_\delta &:= \left\{x : x \in \overline{C} + \delta(x)\overline{\mathcal{B}}\right\} \ , \\ \overline{D}_\delta &:= \left\{x : x \in \overline{D} + \delta(x)\overline{\mathcal{B}}\right\} \ . \end{aligned}$$

The notation "$\overline{\mathrm{co}}$" denotes the closed convex hull.

The following facts can be established:

The basin of attraction is (relatively) open. This means that the basin of attraction $\mathcal{R}$ has the form $\mathcal{O} \bigcap \mathcal{X}$ where $\mathcal{O}$ is open in $\mathbb{R}^n$.

LAS is equivalent to a $\mathcal{KLL}$ estimate. In particular, if $\mathcal{A}$ is LAS with basin of attraction $\mathcal{R} = \mathcal{O} \bigcap \mathcal{X}$, then for each continuous function $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ that is positive definite with respect to $\mathcal{A}$ and proper with respect to $\mathcal{O}$, there exists $\beta \in \mathcal{KLL}$ [3] such that for each solution with $x(0,0) \in \mathcal{R}$, we have

$$\omega(x(t,j)) \leq \beta(\omega(x(0,0)), t, j) \quad \forall (t,j) \in \text{dom } x \ .$$

LAS with open basin of attraction $\mathcal{R}$ implies robust LAS with basin of attraction $\mathcal{R}$. The meaning here should be clear from the definitions above. This result can be used as another route to the robustness results established in (Prieur and Astolfi, 2003) for a hybrid control system applied to a class of nonholonomic systems.

LAS with open basin of attraction $\mathcal{R}$ implies a smooth Lyapunov function on $\mathcal{R}$. A smooth function $V : \mathcal{R} \to \mathbb{R}_{\geq 0}$ is a Lyapunov function relative to a function $\omega : \mathcal{R} \to \mathbb{R}_{\geq 0}$ if there exist functions α_1, $\alpha_2 \in \mathcal{K}_\infty$ [4] and $\lambda \in (0,1)$ such that, for all $x \in \mathcal{R}$,

$$\alpha_1(\omega(x)) \leq V(x) \leq \alpha_2(\omega(x))$$

and

$$\begin{aligned} \langle \nabla V(x), w \rangle \ &\leq \ -V(x) \qquad \forall x \in \overline{C}, w \in F(x) \ , \\ V(w) \ &\leq \ \lambda V(x) \qquad \forall x \in \overline{D}, w \in G(x) \ . \end{aligned}$$

For any continuous function ω satisfying a $\mathcal{KLL}$ estimate of the form given above, there exists such a Lyapunov function.

These conditions for a Lyapunov function are restrictive as sufficient conditions. As necessary conditions, they provide a very strong conclusion which should be compared with the statements in (Michel and Hu, 1999).

Additional facts can be mentioned quickly: forward invariance plus uniform convergence implies stability; zero-input LAS implies local input-to-state stability; LAS is robust to slowly varying parameters, etc.

Moreover, LaSalle's invariance principle can be readily extended to hybrid systems, even those that do not have unique solutions. The results here are modeled after those in (Ryan, 1998) for differential inclusions. They generalize those given in (Lygeros *et al.*, 2003) for hybrid systems.

[3] $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KLL}$ if $\beta(\cdot, t, j)$ is nondecreasing, $\beta(s, \cdot, j)$ is nonincreasing, $\beta(s,t,\cdot)$ is nonincreasing and $\lim_{s \to 0^+} \beta(s,t,j) = \lim_{t \to \infty} \beta(s,t,j) = \lim_{j \to \infty} \beta(s,t,j) = 0$.

[4] $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}_\infty$ if α is continuous, zero at zero, strictly increasing and unbounded.

8. CONCLUSION

In this paper, we have discussed the solutions of hybrid systems from the point of view of guaranteeing that asymptotic stability is robust. We described a notion of a hybrid time domain that treats both ordinary time t and the number of jumps j as independent variables over which the state of the hybrid system solution is defined. We mentioned how this approach allows an efficient characterization of the convergence of a sequence of solutions through results on graphical convergence from set-valued analysis. Then, we showed how to regularize hybrid systems so that every sequence of (possibly perturbed) solutions has a subsequence converging to a some solution of the hybrid system. This result enables showing various properties that follow from asymptotic stability. For instance, for an asymptotically stable compact set for a hybrid system: the basin of attraction is (relatively) open, stability plus convergence equals uniform convergence, the asymptotic stability is robust, and smooth Lyapunov functions exist. These results parallel known facts for differential equations that are used extensively in nonlinear control. Hopefully, they will have significant ramifications for hybrid control systems research as well.

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