Chapter 13 – Confidence Interval Estimation

Example 13.1 (p.352) – Demonstrates the difference between Sampling Theory (Chap.11) and Estimation Theory (Chap. 13). Bottle fill of this production process follows a Normal distribution with σ = 0.01 liters. *Sampling theory* pretends we know μ (the mean fill for the machine) and tells us what to expect for our $\bar{x}$ of n = 6 randomly chosen bottles. Here (*Estimation theory*), we instead learn how to set a confidence interval (CI) for μ. Here, $\bar{x}$ = 2.05 liters, and since $\sigma_{\bar{x}} = \frac{0.01}{\sqrt{6}}$ = 0.00408, the 95% CI for μ is:

2.05 ± 1.960*0.00408, or 2.05 ± 0.0080, or (2.0420 , 2.0580)

Had we wanted a 99% CI for μ, we would obtain:

2.05 ± 2.576*0.00408, or 2.05 ± 0.0105, or (2.0395 , 2.0605)

And a 90% CI for μ is 2.05 ± 1.645*0.00408, or (2.0433 , 2.0567)

Note that the above CI's are predicated upon two very important assumptions: (1) that the parent population (of soda fills from this machine) is Normal, and (2) that σ is known and is known to equal 0.01. If we still keep the Normality assumption, but had we not known σ and obtained a *sample SD* of s = 0.01 ounces instead, we could find a 95% T-distribution CI for μ in the following manner:

2.05 ± 2.5706*0.00408, or 2.05 ± 0.0105, or (2.0395 , 2.0605)

Additional Notes:

(a) The 95% T interval is wider than the 95% Z interval, reflecting the uncertainty about σ

(b) The T interval uses $s_{\bar{x}} = \frac{s}{\sqrt{n}}$ in place of $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$ (Z interval)

(c) The T interval uses t_5 = 2.5706 in place of z = 1.96.

Understanding and interpreting CI's – The text hints at a simulation study on pp.354-5 to help us understand the meaning of a given CI – let's look instead at today's H/O. 100,000 samples of size n = 6 are taken from a Normal distribution with $\mu = 2.03$ and $\sigma = 0.01$; the histograms on p.1 show that the sample means look Normal (but the sample SD's have a right skew). More importantly, for each sample, the corresponding 100,000 95% Z CI's and 95% T CI's are obtained and checked to see whether or not they contain the true value of μ = 2.03. In this simulation, 95.021% of the Z CI's contain the true value, and 95.047% of the T CI's contain the true value. ***This is the correct way to understand confidence intervals – it is incorrect to say that there is a 95% chance that any one interval contains μ.***

One-sided and two-sided CI's – Although the symmetric two-sided CI's are the shortest ones with the given nominal coverage, confidence intervals can also be ***one tailed***. Had we wanted a 95% Z CI above of the form **(a, ∞)**, we would obtain:

a = 2.05 - 1.645*0.00408 = 2.0433, so CI is $(2.0433, \infty)$

Not surprisingly, the value of 'a' in this one-sided 95% CI is the left endpoint of the 90% CI given above.

Choosing the sample size – The margin of error (ME; "**within**") is defined as ME = $z_{1-\alpha/2}\dfrac{\sigma}{\sqrt{n}}$ (bottom of p.368), so given that we know α and σ, we can find n (sample size) to keep ME below some threshold (E). Just choose $n \ge \left(\dfrac{z_{1-\alpha/2}\sigma}{E}\right)^2$. For example (pp. 367-8), for $\alpha = 5\%$ and $\sigma = 136.6$, to get the ME no larger than E = 25 hours ("**within 25 hours of the mean**"), we need a sample size of

$$n \ge \left(\frac{z_{1-\alpha/2}\sigma}{E}\right)^2 = \left(\frac{1.96*136.6}{25}\right)^2 = 114.7 \rightarrow n = 115.$$

<u>Confidence Intervals for a Binomial π</u> – Here, when n is large*, the 95% CI for π is

$$p \pm 1.960 * SE_p \qquad (*)$$

where p = y/n, and $SE_p = \sqrt{\frac{p(1-p)}{n}}$. For CI's of other levels (α's), we would then use other Z values in place of '1.960' in equation (*).

> **<u>Aside</u>: This CI is used instead of the *more accurate CI*, $p \pm 1.960 * \sigma_\pi$ with $\sigma_\pi = \sqrt{\frac{\pi(1-\pi)}{n}}$, since obviously we don't know π in this formula. But, this more accurate CI suggests a so-called quadratic CI or Score CI for π (referenced on p.370); our point for raising this is that there is usually not only one method for finding a CI, although we will just use the method in equation (*).**

Equation (*) simply substitutes p for π in the more accurate CI approach, and since this could be a big mistake, a *more conservative* idea is to put ½ in place of π. When we do so, the 95% CI for p is:

$$p \pm \frac{1.960}{2\sqrt{n}} \qquad (**)$$

Also, an upper bound (E) for the margin of error (within) is therefore $\frac{1.960}{2\sqrt{n}}$, so to get the margin of error at most E, we choose $n \geq \left(\frac{1.96}{2E}\right)^2$

<u>Example 13.5</u>. Here, y = 40 of n = 144 caught fish are bass, which gives p = 27.78% bass. Using the usual approach (*), SE_p = 0.0373, and the 95% CI of 0.2778 ± 1.960*0.0373 or <u>(0.2046 , 0.3509)</u>: we're 95% confident that the true percentage of bass in the lake is between 20.46% and 35.09%. Using the conservative approach (**), we get the 95% CI: 0.2778 ± 0.0817 or <u>(0.1961 , 0.3594)</u>, which is a little wider (more conservative). Finally, how large a sample size is needed to get the margin of error of the 95% CI at most 3%? Answer: 1068 fish.

→ Skip Section 13.7.